Properties of $\Re_m^a$-operator in Minimal Structure Spaces with Ideals

YUTTHAPONG MANUTTIPAROM

A Thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics at Mahasarakham University

March 2020

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The examining committee has unanimously approved this thesis, submitted by Mr. Yutthapong Manuttiparom, as a partial fulfillment of the requirements for the Master of Science in Mathematics at Mahasarakham University.

Examining Committee

....	Chairman
(Asst. Prof. Sommuek Warawiset, Dr.rer.nat.)	(External expert)
....	Committee
(Asst. Prof. Daruni Boonchari, Ph.D.)	(Advisor)
....	Committee
(Asst. Prof. Chokchai Viriyapong, Ph.D.)	(Co-advisor)
....	Committee
(Asst. Prof. Chawalit Boonpok, Ph.D.)	(Faculty graduate committee)
....	Committee
(Asst. Prof. Jeeranunt Khampakdee, Ph.D.)	(Faculty graduate committee)

Mahasarakham University has granted approval to accept this thesis as a partial fulfillment of the requirements for the Master of Science in Mathematics.

....	....
(Prof. Pairot Pramual, Ph.D.)	(Asst. Prof. Krit Chaimoon, Ph.D.)
Dean of the Faculty of Science	Dean of Graduate School

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Finally, I hope that this research will benefit the development of mathematics and lead to further development of education in other areas.

Yutthapong Manuttiparom

ชื่อเรื่อง สมบัติของตัวดำเนินการ $\Re^a_m$ ในปริภูมิโครงสร้างเล็กสุดที่มีอุดมคติ
ผู้วิจัย นายยุทธพงศ์ มนัสทิพารมณ์
ปริญญา วิทยาศาสตรมหาบัณฑิต **สาขา** คณิตศาสตร์
กรรมการควบคุม ผู้ช่วยศาสตราจารย์ ดร. ดรุณี บุญชารี
ผู้ช่วยศาสตราจารย์ ดร. โชคชัย วิริยะพงษ์
มหาวิทยาลัย มหาวิทยาลัยมหาสารคาม **ปีที่พิมพ์** 2563

บทคัดย่อ

ในการวิจัยนี้ ผู้วิจัยได้นำเสนอเซตเปิด δ-m เซตเปิด a-m ฟังก์ชันเฉพาะที่ a ตัวดำเนินการ $\Re^a_m$ บนปริภูมิโครงสร้างเล็กสุดที่มีอุดมคติ พร้อมทั้งศึกษาสมบัติของฟังก์ชัน และตัวดำเนินการนี้ นอกจากนั้นผู้วิจัยยังได้ศึกษา เซตเปิด a-$\mathscr{I}$ และฟังก์ชันต่อเนื่อง a-$\mathscr{I}$

คำสำคัญ : เซตเปิด δ-m, เซตเปิด a-m , ฟังก์ชันเฉพาะที่ a, ตัวดำเนินการ $\Re^a_m$, เซตเปิด a-$\mathscr{I}$, ฟังก์ชันต่อเนื่อง a-$\mathscr{I}$, ปริภูมิโครงสร้างเล็กสุดที่มีอุดมคติ

TITLE Properties of $\Re_m^a$-operator in Minimal Structure Spaces with Ideals

CANDIDATE Mr. Yutthapong Manuttiparom

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ADVISORS Asst. Prof. Daruni Boonchari, Ph.D.,

Asst. Prof. Chokchai Viriyapong, Ph.D.

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ABSTRACT

In this research, the concepts of δ-m-open sets, a-m-open sets in a minimal structure space with an ideal are introduced. In addition, we have presented a-m-local function and $\Re_m^a$-operator in a minimal structure space with an ideal. We have also studied the properties of the function and this operator. Moreover, we have studied a-$\mathscr{I}$-open sets and a-$\mathscr{I}$-continuous.

Keywords : δ-m-open sets, a-m-open sets, a-m-local function, $\Re_m^a$-operator, a-$\mathscr{I}$-open sets, a-$\mathscr{I}$-continuous, a minimal structure space with an ideal.

CONTENTS

CHAPTER 1

Introduction

1.1 Background

Topology is a spatial mathematics also know as topological space. Topological space is defined as a family of sets. Members of the family are considered open set and have the closure property under any unions (finite unions, countable infinite unions and uncountable infinite unions) and finite intersection [16].

Ideal topology is a topological space that contains the additional structure and it is defined as ideal. Kuratowski [13] had proposed the concept of local function in ideal topological space. Kuratowski's concept of operators plays an important role in defining the ideal set of topological space which has the application of localization theory. In the set of topology is defined by Vaidynathaswamy [22]. Later, Jankovic and JHamlett [9] studied the properties of the ideal set in the topological space and introduced the ideal set in other topological spaces by using the a-local function to define the Kuratowski closure operator in a new topological space.

Then in 1996, H. Maki, J. Umehara and T. Noiri [11] defined the minimal structure and studied the properties of this structure, as m is the family of sets. And is the subset of the power set of X, will call (X, m) as the minimal structure space only when there is an empty set and X is a member of m.

In 1945, Vaidyanathaswamy [21] defined the local function in the ideal topological space and studied the properties of this function. Then 2010, Khan M. and Noiri T [12]. defined the semi-local functions in ideal topological space, and also studied properties of such functions.

In 2014, Al-Omeri, along with team [2], has defined the a-local function in the ideal topological space and also studied properties of a-local function.

Later in 2016, Al-Omeri, along with the team [3], defined the $\Re_a$-operator in ideal topological space and studied the properties of this operator.

Therefore, the researcher is interested in studying the a-m-local functions and the $\Re_m^a$-operators in minimal structure spaces, along with studying some properties related to the a-m-local function and the $\mathfrak{R}^a_m$-operator defined above.

1.2 Objective of the research

The purposes of the research are:

1. To define and study some properties of a-m-local function in minimal structure spaces with ideals.
2. To define and study some properties of $\mathfrak{R}_a$-operator in minimal structure spaces with ideals.

1.3 Research process

1. Study and research related documents, reference documents.
2. Write a thesis outline and ask the thesis advisor to check and apply for improvement.
3. Presenting the outline to the thesis examination committee.
4. Improve, edit the layout and present to the graduate school.
5. Study thesis.
6. Present the thesis to the advisor.
7. Organize thesis books.
8. Thesis examination.

1.4 Expected results from this research

1. Define a-m-local function in minimal structure spaces with ideals.
2. Define $\mathfrak{R}^a_m$-operator in minimal structure spaces with ideals.
3. Some properties of a-m-local function in minimal structure spaces with ideals.
4. Some properties of $\mathfrak{R}^a_m$-operator in minimal spaces structure spaces with ideals.

CHAPTER 2

Preliminaries

In this chapter, we will give some definitions, notations, dealing with some preliminaries and some useful results that will be duplicated in later chapter.

2.1 Topological Spaces

The essential properties were distilled out and the concept of a collection of open sets, called a topology, evolved into the following definition:

Definition 2.1.1. [1] Let X be a set. A *topology* τ on X is a collection of subsets of X, each called an *open set*, such that

1. $\emptyset$ and X are open sets.
2. The intersection of finitely many open sets is an open set.
3. The union of any collection of open sets is an open set.

The set X together with a topology τ on X is called a *topological space*, denote by (X, τ).

Thus a collection of subsets of a set X is a topology on X if it includes the empty set and X, and if finite intersections and arbitrary unions of sets in the collection are also in the collection.

Definition 2.1.2. [1] Let X be a set and β be a collection of subsets of X. We say β is a *basis* (for a topology on X) if the following statements hold:

1. For each x in X, there is a B in β such that $x \in B$.
2. If B_1 and B_2 are in β and $x \in B_1 \cap B_2$, then there exists B_3 in β such that $x \in B_3 \subseteq B_1 \cap B_2$.

Definition 2.1.3. [1] A subset A of topological space X is *closed* if the set $X \setminus A$ is open.

Theorem 2.1.4. [1] Let X be a topological space. The following statements about the collection of closed sets in X hold:

1. $\emptyset$ and X are closed.
2. The intersection of any collection of closed sets is a closed set.
3. The union of finitely many closed sets is a closed set.

Definition 2.1.5. [1] Let A be a subset of a topological space X. The *interior* of A, denoted $Int(A)$, is the union of all open sets contained in A. The *closure* of A, denote $Cl(A)$, is the intersection of all closed sets containing A.

Clealy, the interior of A is open and a subset of A, and the closure of A is closed and contain A. Thus we have the aforementioned set sandwich, with A caught between an open set and a closed set: $Int(A) \subseteq A \subseteq Cl(A)$.

Theorem 2.1.6. [1] Let X be a topological space and A and B be subsets of X.

1. If U is an open set in X and $U \subseteq A$, then $U \subseteq Int(A)$.
2. If C is an closed set in X and $A \subseteq C$, then $Cl(A) \subseteq C$.
3. If $A \subseteq B$ then $Int(A) \subseteq Int(B)$.
4. If $A \subseteq B$ then $Cl(A) \subseteq Cl(B)$.
5. A is open if and only if $A = Int(A)$.
6. A is closed if and only if $A = Cl(A)$.

Theorem 2.1.7. [1] For sets A and B in a topological space X, the following statements hold:

1. $Int(X \setminus A) = X \setminus Cl(A)$.
2. $Cl(X \setminus A) = X \setminus Int(A)$.
3. $Int(A) \cup Int(B) \subseteq Int(A \cup B)$, and in general equality does not hold.
4. $Int(A) \cap Int(B) = Int(A \cap B)$.

Definition 2.1.8. [9] An *ideal* $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies the following conditions:

1. $A \in \mathscr{I}$ and $B \subseteq A$ implies $B \in \mathscr{I}$.

2. $A \in \mathscr{I}$ and $B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$.

Definition 2.1.9. [19] Let X be a topological space and A be subset of X.

1. A is said to be *regular open* if $A = Int(Cl(A))$.

2. A is said to be *regular closed* if $A = Cl(Int(A))$.

2.2 Local Function

Definition 2.2.1. [9, 21] An ideal topological space is a topological space (X, τ) with an ideal $\mathscr{I}$ on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \longrightarrow P(X)$, called a *local function* of A with respect to τ and $\mathscr{I}$ is defined as follows: for $A \subseteq X$, $A^*(\mathscr{I}, \tau) = \{x \in X : U \cap A \notin \mathscr{I}$, for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$.

Lemma 2.2.2. [21] Let $(X, \tau, \mathscr{I})$ be an ideal topological space, and $A, B \subseteq X$. Then the following properties hold:

1. If $A \subseteq B$, then $A^* \subseteq B^*$.

2. If $U \in \tau$, then $U \cap A^* \subseteq (U \cap A)^*$.

3. $A^* = Cl(A^*) \subseteq Cl(A)$.

4. $(A \cup B)^* = A^* \cup B^*$.

5. $(A \cap B)^* \subseteq A^* \cup B^*$.

2.3 a-Local Function

Definition 2.3.1. [22] Let (X, τ) be a topological space and $A \subseteq X$.

1. A is called δ-*open* if for each $x \in A$, there exists a regular open set G such that $x \in G \subseteq A$. The complement of δ-open is called δ-*closed*. The collection of all δ-open sets in X is denoted by $\delta O(X)$.

2. A point $x \in X$ is called a δ-*cluster* point of A if $Int(Cl(U)) \cap A \neq \emptyset$ for each open set U containing x.

3. The set of all δ-cluster points of A is called the δ-*closure* of A and is denoted by $Cl_\delta(A)$.

4. The set δ-*interior* of A is denoted by $Int_\delta(A)$, $Int_\delta(A) = \cup\{U : U$ is a regular open set, $U \subseteq A\}$.

Theorem 2.3.2. [22] Let (X, τ) be a topological space and $A \subseteq X$. A is δ-open if and only if $Int_\delta(A) = A$.

Definition 2.3.3. [5, 6] Let (X, τ) be a topological space and $A \subseteq X$.

1. A is said to be a-*open* if $A \subseteq Int(Cl(Int_\delta(A)))$.

2. A is said to be a-*closed* if $Cl(Int(Cl_\delta(A))) \subseteq A$.

3. The family of all a-open sets of X, denoted by τ^a.

4. The collection of all a-open sets containing x in X is denoted by $\tau^a(x)$.

5. The intersection of all a-closed sets containing A is called a-*closure* of A and is denoted by $aCl(A)$.

6. The a-*interior* of A, denoted by $aInt(A)$, is defined by the union of all a-open sets contained in A.

Definition 2.3.4. [2] Let $(X, \tau, \mathscr{I})$ be an ideal topological space and A be a subset of X. Then $A^{a^*}(\mathscr{I}, \tau) = \{x \in X : U \cap A \notin \mathscr{I}$, for every $U \in \tau^a(x)\}$ is called a-*local function* of A with respect to $\mathscr{I}$ and τ. We denote simply A^{a^*} for $A^{a^*}(\mathscr{I}, \tau)$.

Remark 2.3.5. [2]

1. The minimal ideal is $\{\emptyset\}$ in any ideal topological space $(X, \tau, \mathscr{I})$ and the maximal ideal is $P(X)$. It can be deduced that $A^{a^*}(\{\emptyset\}) = aCl(A) \neq Cl(A)$ and $A^{a^*}(P(X)) = \emptyset$ for every $A \subseteq X$.

2. If $A \in \mathscr{I}$, then $A^{a^*} = \emptyset$.

3. Neither $A \subseteq A^{a^*}$ nor $A^{a^*} \subseteq A$ in general.

Theorem 2.3.6. [2] Let $(X, \tau, \mathscr{I})$ be an ideal topological space and A, B be subsets of X. Then for a-local functions the following properties hold:

1. $\tau^a \cap \mathscr{I} = \emptyset$.
2. If $J \in \mathscr{I}$ then $aInt(J) = \emptyset$.
3. For every $G \in \tau^a$ then $G \subseteq G^{a^*}$.
4. $X = X^{a*}$.

Theorem 2.3.7. [2] Let $(X, \tau, \mathscr{I})$ be an ideal topological space and A, B subsets of X. Then for a-local function the following properties hold:

1. $(\emptyset)^{a^*} = \emptyset$.
2. If $A \subseteq B$, then $A^{a^*} \subseteq B^{a^*}$.
3. For another ideal $\mathscr{I} \subseteq \mathscr{J}$ on X, $A^{a^*}(\mathscr{J}, \tau) \subseteq A^{a^*}(\mathscr{I}, \tau)$.
4. $A^{a^*} \subseteq aCl(A)$.
5. $A^{a^*}(\mathscr{I}, \tau) = aCl(A^{a^*}) \subseteq aCl(A)$ (i.e A^{a^*} is an a-closed subset of $aCl(A)$).
6. $(A^{a^*})^{a^*} \subseteq A^{a^*}$.
7. $(A \cup B)^{a^*} = A^{a^*} \cup B^{a^*}$.
8. $A^{a^*} \setminus B^{a^*} = (A \setminus B)^{a^*} \setminus B^{a^*} \subseteq (A \setminus B)^{a^*}$.
9. If $U \in \tau^a$, then $U \cap A^{a^*} = U \cap (U \cap A)^{a^*} \subseteq (U \cap A)^{a^*}$.
10. If $U \in \mathscr{I}$, then $(A \setminus U)^{a^*} \subseteq A^{a^*} = (A \cup U)^{a^*}$.
11. If $A \subseteq A^{a^*}$, then $A^{a^*}(\mathscr{I}, \tau) = aCl(A^{a^*}) = aCl(A)$.

Theorem 2.3.8. [2] Let (X, τ) be a topological space, $\mathscr{I}$ and $\mathscr{J}$ be ideals on X and let A be a subset of X. Then the following properties hold:

1. If $\mathscr{I} \subseteq \mathscr{J}$, then $A^{a^*}(\mathscr{J}, \tau) \subseteq A^{a^*}(\mathscr{I}, \tau)$.

2. If $A^{a^*}((\mathscr{I} \cap \mathscr{J}), \tau) = A^{a^*}(\mathscr{I}, \tau) \cup A^{a^*}(\mathscr{J}, \tau)$.

Lemma 2.3.9. [2] Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If $U \in \tau^a(x)$, then $U \cap A^{a^*} = U \cap (U \cap A)^{a^*} \subseteq (U \cap A)^{a^*}$ for any subset A of X.

Definition 2.3.10. [2] Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then τ is said to be *a-compatible* with respect to $\mathscr{I}$, denoted by $\mathscr{I} \sim^a \tau$ if and only if, for every $x \in A$ there exists $U \in \tau^a(x)$ such that $U \cap A \in \mathscr{I}$, then $A \in \mathscr{I}$.

Theorem 2.3.11. [2] Let $(X, \tau, \mathscr{I})$ be an ideal topological space and A a subset of X. Then the following are equivalent:

1. $\mathscr{I} \sim^a \tau$.
2. If a subset A of X has a cover a-open sets of whose intersection with A is in $\mathscr{I}$, then A is in $\mathscr{I}$.
3. For every $A \subseteq X$, if $A \cap A^{a^*} = \emptyset$, $A \in \mathscr{I}$.
4. For every $A \subseteq X$, $A \setminus A^{a^*} \in \mathscr{I}$.
5. For every $A \subseteq X$, if A contains no nonempty subset B with $B \subseteq B^{a^*}$, then $A \in \mathscr{I}$.

Theorem 2.3.12. [2] Let $(X, \tau, \mathscr{I})$ be an ideal topological space and A a subset of X. if τ is a-compatible with $\mathscr{I}$, then the following are equivalent:

1. For every $A \subseteq X$, if $A \cap A^{a^*} = \emptyset$ implies $A^{a^*} = \emptyset$.
2. For every $A \subseteq X$, $(A \setminus A^{a^*})^{a^*} = \emptyset$.
3. For every $A \subseteq X$, $(A \cap A^{a^*})^{a^*} = A^{a^*}$.

2.4 On $\Re_a$-Operator In Ideal Topological Spaces

In this section we shall introduce the operator $\Re_a$ in $(X, \tau, \mathscr{I})$. Kuratowski has shown that $Cl(A) = X \setminus Int(X \setminus A)$. This relation is the motivation of defining the operator $\Re_a$. We shall also discuss the behaviour of this operator.

Definition 2.4.1. [3] Let $(X, \tau, \mathscr{I})$ be an ideal topological space. An operator $\Re_a : P(X) \to \tau^a$ is defined as follows: for every $A \in P(X)$,

$$\Re_a(A) = \{x \in X : \text{there exists } U_x \in \tau^a \text{ such that } U_x \setminus A \in \mathscr{I}\}.$$

Theorem 2.4.2. [3] Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then for $A \in P(X), \Re_a(A) = X \setminus (X \setminus A)^{a^*}$.

Theorem 2.4.3. [3] Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then the following properties hold:

1. If $A \subseteq X$, then $\Re_a(A)$ is a-open.

2. If $A \subseteq B$, then $\Re_a(A) \subseteq \Re_a(B)$.

3. If $A, B \in P(X)$, then $\Re_a(A \cap B) = \Re_a(A) \cap \Re_a(B)$.

4. If $U \in \tau^{a^*}$, then $U \subseteq \Re_a(U)$.

5. If $A \subseteq X$, then $\Re_a(A) \subseteq \Re_a(\Re_a(A))$.

6. If $A \subseteq X$, then $\Re_a(A) = \Re_a(\Re_a(A))$ if and only if $(X \setminus A)^{a^*} = ((X \setminus A)^{a^*})^{a^*}$.

7. If $A \in \mathscr{I}$, then $\Re_a(A) = X \setminus X^{a^*}$.

8. If $A \subseteq X$, then $A \cap \Re_a(A) = Int^{a^*}(A)$, where Int^{a^*} is the interior of τ^{a^*}.

9. If $A \subseteq X, J \in \mathscr{I}$, then $\Re_a(A \setminus J) = \Re_a(A)$.

10. If $A \subseteq X, J \in \mathscr{I}$, then $\Re_a(A \cup J) = \Re_a(A)$.

11. If $(A \setminus B) \cup (B \setminus A) \in \mathscr{I}$, then $\Re_a(A) = \Re_a(B)$.

12. If $A, B \in P(X)$, then $\Re_a(A \cap B) \subseteq \Re_a(A) \cup \Re_a(B)$.

Corollary 2.4.4. [3] Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then $U \subseteq \Re_a(U)$ for every a-open set U.

Theorem 2.4.5. [3] Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \subseteq X$. Then the following properties hold:

1. $\Re_a(A) = \{x \in X : \text{there exists } U_x \in \tau^a(x) : U_x \setminus A \in \mathscr{I}\}$.

2. $\Re_a(A) = \cup\{U \in \tau^a : U \setminus A \in \mathscr{I}\}$.

Remark 2.4.6. [3] Let $\mathscr{I} = \{\emptyset\}$, then $\Re_a(A) = \cup\{U \in \tau^a : U \setminus A = \emptyset\} = \cup\{U \in \tau^a : U \subseteq A\} = aInt(A)$, for any space (X, τ).

Theorem 2.4.7. [3] Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \subseteq X$. Then for any a-open set A of X, $\Re_a(A) = \cup\{U \in \tau^a : (U \setminus A) \cup (A \setminus U) \in \mathscr{I}\}$.

2.5 Minimal Structure

Definition 2.5.1. [17] Let X be a nonempty set and $P(X)$ the power set of X. A subfamily m of $P(X)$ is called a $minimal\ structure$ (briefly MS) on X if $\emptyset \in m$ and $X \in m$.

By (X, m), we donote a nonempty set X with a minimal structure m on X and it is called a minimal structure space. Each member of m is said to be m-*open* and the complement of m-open is said to be m-*closed*.

Definition 2.5.2. [17] Let (X, m) be a minimal structure space and $A \subseteq X$, the m-*closure* of A denoted by $Cl_m(A)$ and the m-*interior* of A denoted by $Int_m(A)$, are defined as follows:

1. $Cl_m(A) = \bigcap\{F : A \subseteq F, X \setminus F \in m\}$.

2. $Int_m(A) = \bigcup\{U : U \subseteq A, U \in m\}$.

Lemma 2.5.3. [14] Let (X, m) be a minimal structure space and $A, B \subseteq X$, the following properties hold:

1. $Cl_m(X \setminus A) = X \setminus Int_m(A)$ and $Int_m(X \setminus A) = X \setminus Cl_m(A)$.

2. if $X \setminus A \in m$, then $Cl_m(A) = A$ and if $A \in m$, then $Int_m(A) = A$.

3. $Cl_m(\emptyset) = \emptyset$, $Cl_m(X) = X$, $Int_m(\emptyset) = \emptyset$ and $Int_m(X) = X$.

4. if $A \subseteq B$, then $Cl_m(A) \subseteq Cl_m(B)$ and $Int_m(A) \subseteq Int_m(B)$.

5. $A \subseteq Cl_m(A)$ and $Int_m(A) \subseteq A$.

6. $Cl_m(Cl_m(A)) = Cl_m(A)$ and $Int_m(Int_m(A)) = Int_m(A)$.

Lemma 2.5.4. [14] Let (X, m) be a minimal structure space and $A \subseteq X$. Then $x \in Cl_m(A)$ if and only if $U \cap A \neq \emptyset$ for every an m-open set U containing x.

Definition 2.5.5. [7] Let (X, m) be a minimal structure sapce and $A \subseteq X$.

1. A is called *m-regular open* if $A = Int_m(Cl_m(A))$.

2. A is called *m-regular closed* if $X \setminus A$ is m-regular open.

The family of all m-regular open sets of X denoted by $r(m)$ and the family of all m-regular closed sets of X denoted by $rc(m)$.

Definition 2.5.6. [18] An ideal $\mathscr{I}$ on a minimal structure space (X, m) is a nonempty collection of subsets of X which satisfies the following properties;

1. $A \in \mathscr{I}$ and $B \subseteq A$ implies $B \in \mathscr{I}$. (heredity)

2. $A \in \mathscr{I}$ and $B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$. (finite additivity)

The set $\mathscr{I}$ together with a minimal structure space (X, m) is called a *minimal structure space with an ideal*, denote by $(X, m, \mathscr{I})$.

CHAPTER 3

a-m-Local Function and $\Re_m^a$-Operator

In this section, we introduce the concepts of δ-m-open sets and a-m-open sets in a minimal structure space with an ideal and study some fundamental properties.

Moreover, we introduced notion of a-m-local function and $\Re_m^a$-operator in a minimal structure space with an ideal. Some properties of their are obtained.

3.1 δ-m-open set and a-m-open set

Definition 3.1.1. Let (X, m) be a minimal structure space and $A \subseteq X$.

1. A is called *δ-m-open* if for each $x \in A$ there exists an m-regular open set G such that $x \in G \subseteq A$. The family of all δ-m-open sets of X denoted by $\delta O_m(X)$.

2. The complement of δ-m-open is called *δ-m-closed*. The family of all δ-m-closed sets of X denoted by $\delta C_m(X)$.

3. A point $x \in X$ is called a *δ-m-cluster point* of A if $U \cap A \neq \emptyset$ for each m-regular open set U containing x.

4. The set of all δ-m-cluster points of A is called the *δ-m-closure* of A and is denoted by $C_{\delta m}(A)$.

5. The set *δ-m-interior* is denoted by $I_{\delta m}(A) = \bigcup\{U : U$ is m-regular open and $U \subseteq A\}$.

Theorem 3.1.2. The arbitrary union of δ-m-open sets is an δ-m-open set.

Proof. Let B_α be an δ-m-open set for all $\alpha \in J$ where J is an index set and let $x \in \bigcup_{\alpha \in J} B_\alpha$. There exists $\beta \in J$ such that $x \in B_\beta$. Since B_β is δ-m-open, there exists m-regular open G_β such that $x \in G_\beta \subseteq B_\beta$. Then $x \in G_\beta \subseteq B_\beta \subseteq \bigcup_{\alpha \in J} B_\alpha$. Therefore $\bigcup_{\alpha \in J} B_\alpha$ is δ-m-open. $\square$

Theorem 3.1.3. Let (X, m) be a minimal structure space and $A \subseteq X$. Then A is δ-m-open if and only if $I_{\delta m}(A) = A$.

Proof. ($\Longrightarrow$) Suppose that A is δ-m-open. By definition of δ-m-interior, $I_{\delta m}(A) \subseteq A$.

Let $x \in A$. Since A is δ-m-open, there exists m-regular open O such that $x \in O \subseteq A$. This implies that $x \in I_{\delta m}(A)$. Then $A \subseteq I_{\delta m}(A)$. Hence $A = I_{\delta m}(A)$.

($\Longleftarrow$) It follows from Theorem 3.1.2. □

Theorem 3.1.4. Let (X, m) be a minimal structure space and $A, B \subseteq X$. The following property hold;

1. If $A \subseteq B$, then $I_{\delta m}(A) \subseteq I_{\delta m}(B)$.
2. If $A \subseteq B$, then $C_{\delta m}(A) \subseteq C_{\delta m}(B)$.

Proof. 1. Assume that $A \subseteq B$ and $x \in I_{\delta m}(A)$. Then, there exists m-regular open G such that $x \in G \subseteq A$. Since $A \subseteq B$, we have $x \in G \subseteq A \subseteq B$. This implies that $x \in I_{\delta m}(B)$. Hence $I_{\delta m}(A) \subseteq I_{\delta m}(B)$.

2. Let $A \subseteq B$. Assume that $x \notin C_{\delta m}(B)$. Then there exists m-regular open U containing x such that $U \cap B = \emptyset$. Since $A \subseteq B$, we have $U \cap A \subseteq U \cap B = \emptyset$. Thus $x \notin C_{\delta m}(A)$. Therefore $C_{\delta m}(A) \subseteq C_{\delta m}(B)$. □

Theorem 3.1.5. Let (X, m) be a minimal structure space and $A \subseteq X$. The following property hold;

1. $C_{\delta m}(A) = X \setminus I_{\delta m}(X \setminus A)$.
2. $I_{\delta m}(A) = X \setminus C_{\delta m}(X \setminus A)$.

Proof. 1. We will show that $C_{\delta m}(A) \subseteq X \setminus I_{\delta m}(X \setminus A)$. Assume that $C_{\delta m}(A) \nsubseteq X \setminus I_{\delta m}(X \setminus A)$. Then there exists $x \in C_{\delta m}(A)$ such that $x \notin X \setminus I_{\delta m}(X \setminus A)$. We get that $x \in I_{\delta m}(X \setminus A)$. So there exists m-regular open G such that $x \in G \subseteq X \setminus A$, we get that $G \cap (X \setminus A) = G$. Then $G \cap A = (G \cap (X \setminus A)) \cap A = G \cap ((X \setminus A) \cap A) = G \cap \emptyset = \emptyset$. Since $x \in C_{\delta m}(A)$, $G \cap A \neq \emptyset$. It is contradiction. Thus $C_{\delta m}(A) \subseteq X \setminus I_{\delta m}(X \setminus A)$.

Next, we will show that $X \setminus I_{\delta m}(X \setminus A) \subseteq C_{\delta m}(A)$. Assume $x \in X \setminus I_{\delta m}(X \setminus A)$

and $x \notin C_{\delta m}(A)$. Since $x \in X \setminus I_{\delta m}(X \setminus A)$, then $x \notin I_{\delta m}(X \setminus A)$. Since $x \notin C_{\delta m}(A)$, then x is not δ-m-cluster point of A. There exists m-regular open G containing x such that $G \cap A = \emptyset$. So $x \in G \subseteq X \setminus A$, we get that $x \in I_{\delta m}(X \setminus A)$, which contradicts with $x \notin I_{\delta m}(X \setminus A)$. Thus $X \setminus I_{\delta m}(X \setminus A) \subseteq C_{\delta m}(A)$.

2. Since $X \setminus A \subseteq X$, then by 1, we have $C_{\delta m}(X \setminus A) = X \setminus I_{\delta m}(X \setminus (X \setminus A))$, we get $C_{\delta m}(X \setminus A) = X \setminus I_{\delta m}(A)$. Therefore $I_{\delta m}(A) = X \setminus C_{\delta m}(X \setminus A)$. □

Definition 3.1.6. Let (X, m) be a minimal structure space and $A \subseteq X$.

1. A is called *a-m-open* if $A \subseteq Int_m(Cl_m(I_{\delta m}(A)))$. The family of all a-m-open sets of X denoted by $\mathscr{M}^a$.

2. A is called *a-m-closed* if $Cl_m(Int_m(C_{\delta m}(A))) \subseteq A$.

3. The family of all a-m-open sets containing x in X is denoted by $\mathscr{M}^a(x)$.

Definition 3.1.7. Let (X, m) be a minimal structure space and $A \subseteq X$, the *a-m-closure* of A denoted by $aC_m(A)$ and the *a-m-interior* of A denoted by $aI_m(A)$, are defined as follows:

1. $aC_m(A) = \bigcap\{F : X \setminus F \in \mathscr{M}^a \text{ and } A \subseteq F\}$

2. $aI_m(A) = \bigcup\{U : U \in \mathscr{M}^a \text{ and } U \subseteq A\}$.

Example 3.1.8. Let $X = \{a, b, c, d\}$ with a minimal structure
$m = \{\emptyset, \{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, X\}$.
Then $r(m) = \{\emptyset, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, X\}$,
and $\delta O_m(x) = \{\emptyset, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$,
$\mathscr{M}^a = \{\emptyset, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

In this example $\{a, b\}$, $\{a, d\} \in \mathscr{M}^a$ but $\{a, b\} \cap \{a, d\} = \{a\} \notin \mathscr{M}^a$. That is, $\mathscr{M}^a$ doesn't have the property that any finite intersection of a-m-open sets is a-m-open.

Theorem 3.1.9. Let (X, m) be a minimal structure space and $A \subseteq X$. Then $x \in aC_m(A)$ if and only if $U \cap A \neq \emptyset$ for every a-m-open set U containing x.

Proof. ($\Longrightarrow$) Suppose that U be a-m-open containing x such that $U \cap A = \emptyset$. So $A \subseteq X \setminus U$ and $X \setminus U$ is a-m-closed. Since $aC_m(A)$ is the intersection of all a-m-closed sets containing A, then $aC_m(A) \subseteq X \setminus U$. Since $x \notin X \setminus U$, we have $x \notin aC_m(A)$.

($\Longleftarrow$) Assume that $x \notin aC_m(A)$. Then there exists a-m-closed F such that $A \subseteq F$ and $x \notin F$. Choose $U = X \setminus F$. Then $U = X \setminus F$ is a-m-open and $x \in X \setminus F = U$. Moreover, $U \cap A = (X \setminus A) \cap A = \emptyset$. □

Theorem 3.1.10. Let (X, m) be a minimal structure space and $A, B \subseteq X$. The following property hold;

1. If $A \subseteq B$, then $aC_m(A) \subseteq aC_m(B)$.
2. If $A \subseteq B$, then $aI_m(A) \subseteq aI_m(B)$.

Proof. 1. Let $A \subseteq B$ and $x \notin aC_m(B)$. Then there exists a-m-open U containing x such that $U \cap B = \emptyset$. Since $A \subseteq B$. So $U \cap A = \emptyset$. Hence $x \notin aC_m(A)$.

2. Let $A \subseteq B$ and $x \in aI_m(A)$. Then there exists a-m-open U such that $x \in U \subseteq A$. Since $A \subseteq B$, $x \in U \subseteq B$. Therefore $x \in aI_m(B)$. □

Proposition 3.1.11. Let (X, m) be a minimal structure space. Then $\emptyset \in \mathscr{M}^a$ and $X \in \mathscr{M}^a$.

Proof. Since $\emptyset \subseteq Int_m(Cl_m(I_{\delta m}(\emptyset)))$. Then $\emptyset$ is a-m-open, and so $\emptyset \in \mathscr{M}^a$.

Clearly $X = Int_m(Cl_m(X))$, so X is m-regular open set. Then X be δ-m-open too, that is $I_{\delta m}(X) = X$, $X \subseteq Int_m(Cl_m(I_{\delta m}(X)))$. Therefore $X \in \mathscr{M}^a$. □

Theorem 3.1.12. Let (X, m) be a minimal structure space, then the arbitrary union of elements of $\mathscr{M}^a$ belong to $\mathscr{M}^a$.

Proof. Let V_α be a-m-open for all $\alpha \in J$ and $G = \bigcup_{\alpha \in J} V_\alpha$. Then $V_\alpha \subseteq Int_m(Cl_m(I_{\delta m}(V_\alpha)))$ for all $\alpha \in J$. Since $V_\alpha \subseteq G$, $I_{\delta m}(V_\alpha) \subseteq I_{\delta m}(G)$, and so $Cl_m(_{\delta m}(V_\alpha)) \subseteq Cl_m(I_{\delta m}(G))$. Then $Int_m(Cl_m(I_{\delta m}(V_\alpha))) \subseteq Int_m(Cl_m(I_{\delta m}(G)))$. This implies that $V_\alpha \subseteq Int_m(Cl_m(I_{\delta m}(G)))$ for all $\alpha \in J$. Thus $\bigcup_{\alpha \in J} V_\alpha \subseteq Int_m(Cl_m(I_{\delta m}(G)))$. Therefore $G \subseteq Int_m(Cl_m(I_{\delta m}(G)))$. □

Lemma 3.1.13. Let (X, m) be a minimal structure space. Then the arbitrary intersection of a-m-closed sets is an a-m-closed set.

Proof. Let G_α be a-m-closed for all $\alpha \in J$. Then $X \setminus G_\alpha$ is a-m-open and so $\bigcup_{\alpha\in J}(X\setminus G_\alpha)$ is a-m-open. Since $X\setminus\bigcap_{\alpha\in J} G_\alpha = \bigcup_{\alpha\in J}(X\setminus G_\alpha)$, $\bigcap_{\alpha\in J} G_\alpha$ is a-m-closed. □

Remark 3.1.14. In a minimal structure space, by Lemma 3.1.13, $aC_m(A)$ is a-m-closed.

Theorem 3.1.15. Let (X, m) be a minimal structure space and $A \subseteq X$. The following property hold;

1. $aC_m(aC_m(A)) = aC_m(A)$.
2. $aI_m(aI_m(A)) = aI_m(A)$.

Proof.
1. Clearly $aC_m(A) \subseteq aC_m(aC_m(A))$. Since $aC_m(A)$ is a-m-closed, then $aC_m(aC_m(A)) \subseteq aC_m(A)$. Therefore $aC_m(aC_m(A)) = aC_m(A)$.
2. Clearly $aI_m(aI_m(A)) \subseteq aI_m(A)$. Since $aI_m(A)$ is a-m-open, then $aI_m(A) \subseteq aI_m(aI_m(A))$. Therefore $aI_m(aI_m(A)) = aI_m(A)$. □

3.2 a-m-Local Function in Minimal Structure Spaces with Ideals

Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $\mathscr{M}^a(x) = \{U : x \in U, U \in \mathscr{M}^a\}$ be the family of a-m-open sets which contain a point $x \in X$.

Definition 3.2.1. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A \subseteq X$. Then

$$A_m^{a^*}(\mathscr{I}, m) = \{x \in X : U \cap A \notin \mathscr{I}, \text{ for every } U \in \mathscr{M}^a(x)\}$$

is called *a-m-local function of A with respect to $\mathscr{I}$ and m*. We denote simply $A_m^{a^*}$ for $A_m^{a^*}(\mathscr{I}, m)$.

Remark 3.2.2.
1. The minimal ideal is $\{\emptyset\}$ and the maximal ideal is $P(X)$ in any minimal structure space with an ideal $(X, m, \mathscr{I})$. It can be deduced that $A_m^{a^*}(\{\emptyset\}, m) = aC_m(A)$ and $A_m^{a^*}(P(X), m) = \emptyset$ for every $A \subseteq X$.
2. $A \nsubseteq A_m^{a^*}$ and $A_m^{a^*} \nsubseteq A$, in general.

Example 3.2.3. Let $X = \{a, b, c, d\}$ with a minimal structure
$m = \{\emptyset, \{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, X\}$ and $\mathscr{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, $A = \{a, b\}$.
Then $\mathscr{M}^a = \{\emptyset, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$
and $A_m^{a^*} = \emptyset$.

Example 3.2.4. Let $X = \{a, b, c, d\}$ with a minimal structure
$m = \{\emptyset, \{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, X\}$ and $\mathscr{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, $A = \{c, d\}$.
Then $\mathscr{M}^a = \{\emptyset, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$
and $A_m^{a^*} = \{c, d\}$.

Theorem 3.2.5. Let $(X, m, \mathscr{I})$ be a minimal structure space with ideal and $A, B \subseteq X$. The following properties hold;

1. $(\emptyset)_m^{a^*} = \emptyset$.
2. If $A \subseteq B$, then $A_m^{a^*} \subseteq B_m^{a^*}$.
3. For another ideal $\mathscr{J}$ on X such that $\mathscr{I} \subseteq \mathscr{J}$ then $A_m^{a^*}(\mathscr{J}, m) \subseteq A_m^{a^*}(\mathscr{I}, m)$.
4. $A_m^{a^*} \subseteq aC_m(A)$.
5. $A_m^{a^*} = aC_m(A_m^{a^*})$, (i.e., $A_m^{a^*}$ is a-m-closed).
6. $(A_m^{a^*})_m^{a^*} \subseteq A_m^{a^*}$.
7. $A_m^{a^*} \cup B_m^{a^*} \subseteq (A \cup B)_m^{a^*}$.
8. $(A \cap B)_m^{a^*} \subseteq A_m^{a^*} \cap B_m^{a^*}$.
9. $(A \setminus B)_m^{a^*} \setminus B_m^{a^*} \subseteq A_m^{a^*} \setminus B_m^{a^*}$.
10. If $A \in \mathscr{I}$, then $A_m^{a^*} = \emptyset$.
11. If $U \in \mathscr{I}$, then $A_m^{a^*} = (A \cup U)_m^{a^*}$.
12. If $U \in \mathscr{I}$, then $A_m^{a^*} = (A \setminus U)_m^{a^*}$.

Proof. 1. Assume $(\emptyset)_m^{a^*} \neq \emptyset$, then there exists $x \in (\emptyset)_m^{a^*}$. Since $X \in \mathscr{M}^a(x)$, $X \cap \emptyset \notin \mathscr{I}$. It is a contradiction with $X \cap \emptyset = \emptyset \in \mathscr{I}$. Therefore $(\emptyset)_m^{a^*} = \emptyset$.

2. Assume that $A \subseteq B$. We will show that $A_m^{a^*} \subseteq B_m^{a^*}$. Suppose that $x \notin B_m^{a^*}$. Then there exists $U \in \mathscr{M}^a(x)$ such that $U \cap B \in \mathscr{I}$. From $A \subseteq B$ and the property of $\mathscr{I}$, $U \cap A \in \mathscr{I}$. Therefore $x \notin A_m^{a^*}$.

3. Assume that $\mathscr{J}$ is ideal such that $\mathscr{I} \subseteq \mathscr{J}$ and $x \in A_m^{a^*}(\mathscr{J}, m)$, then $U \cap A \notin \mathscr{J}$ for every $U \in \mathscr{M}^a(x)$. This implies that $U \cap A \notin \mathscr{I}$ for every $U \in \mathscr{M}^a(x)$, so $x \in A_m^{a^*}(\mathscr{I}, m)$. Hence $A_m^{a^*}(\mathscr{J}, m) \subseteq A_m^{a^*}(\mathscr{I}, m)$.

4. Assume that $x \notin aC_m(A)$. Then there exists a-m-closed F such that $A \subseteq F$ and $x \notin F$. Thus $x \in X \setminus F$, and so $X \setminus F \in \mathscr{M}^a(x)$. Hence $(X \setminus F) \cap A = \emptyset \in \mathscr{I}$, and so $x \notin A_m^{a^*}$. This implies that $A_m^{a^*} \subseteq aC_m(A)$.

5. It is clear that $A_m^{a^*} \subseteq aC_m(A_m^{a^*})$. Next, we will prove that $aC_m(A_m^{a^*}) \subseteq A_m^{a^*}$. Let $x \in aC_m(A_m^{a^*})$ and $U \in \mathscr{M}^a(x)$. Then $A_m^{a^*} \cap U \neq \emptyset$. Therefore there exists $y \in A_m^{a^*} \cap U$, so $U \in \mathscr{M}^a(y)$. Since $y \in A_m^{a^*}, A \cap U \notin \mathscr{I}$, and so $x \in A_m^{a^*}$. Hence we have $aC_m(A_m^{a^*}) \subseteq A_m^{a^*}$ and $aC_m(A_m^{a^*}) = A_m^{a^*}$.

6. Assume that $x \in (A_m^{a^*})_m^{a^*}$, and $U \in \mathscr{M}^a(x)$. Then $A_m^{a^*} \cap U \notin \mathscr{I}$ and so $A_m^{a^*} \cap U \neq \emptyset$. Thus there exists $y \in A_m^{a^*} \cap U$, and so $y \in U \in \mathscr{M}^a(y)$. This implies that $A \cap U \notin \mathscr{I}$. Therefore, $x \in A_m^{a^*}$.

7. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by 2, $A_m^{a^*} \subseteq (A \cup B)_m^{a^*}$ and $B_m^{a^*} \subseteq (A \cup B)_m^{a^*}$. So $A_m^{a^*} \cup B_m^{a^*} \subseteq (A \cup B)_m^{a^*}$.

8. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by 2, $(A \cap B)_m^{a^*} \subseteq A_m^{a^*}$ and $(A \cap B)_m^{a^*} \subseteq B_m^{a^*}$. So $(A \cap B)_m^{a^*} \subseteq A_m^{a^*} \cap B_m^{a^*}$.

9. Since $A \setminus B \subseteq A$ by 2, then $(A \setminus B)_m^{a^*} \subseteq A_m^{a^*}$. So $(A \setminus B)_m^{a^*} \setminus B_m^{a^*} \subseteq A_m^{a^*} \setminus B_m^{a^*}$.

10. Assume that $A_m^{a^*} \neq \emptyset$. Then there exists $x \in A_m^{a^*}$. Since $X \in \mathscr{M}^a(x)$, $A = X \cap A \notin \mathscr{I}$.

11. Assume that $U \in \mathscr{I}$. Since $A \subseteq A \cup U$, by 2, we get $A_m^{a^*} \subseteq (A \cup U)_m^{a^*}$. Next, we will prove that $(A \cup U)_m^{a^*} \subseteq A_m^{a^*}$. Suppose that $x \notin A_m^{a^*}$. Then there exists $V \in \mathscr{M}^a(x)$ such that $A \cap V \in \mathscr{I}$. Since $(A \cup U) \cap V = (A \cap V) \cup (U \cap V) \in \mathscr{I}$, then $(A \cup U) \cap V \in \mathscr{I}$. Therefore $x \notin (A \cup U)_m^{a^*}$.

12. Assume that $U \in \mathscr{I}$. Since $A^{a^*}_m = (A \cap X)^{a^*}_m = (A \cap ((X \setminus U) \cup U))^{a^*}_m = ((A \setminus U) \cup (A \cap U))^{a^*}_m$ and $A \cap U \subseteq U \in \mathscr{I}$ by 11, then $A^{a^*}_m = (A \setminus U)^{a^*}_m$. □

Definition 3.2.6. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal, the *closure operator* of A denoted by $aC^*_m(A)$ where $aC^*_m(A) = A \cup A^{a^*}_m$.

Theorem 3.2.7. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal where $aC^*_m(A) = A \cup A^{a^*}_m$ and $A, B \subseteq X$. Then

1. $aC^*_m(\emptyset) = \emptyset$.
2. $A \subseteq aC^*_m(A)$.
3. $aC^*_m(A) \cup aC^*_m(B) \subseteq aC^*_m(A \cup B)$.
4. $aC^*_m(A) \subseteq aC^*_m(aC^*_m(A))$.

Proof. 1. By Theorem 3.2.5(1), we get that $aC^*_m(\emptyset) = \emptyset \cup (\emptyset)^{a^*}_m = \emptyset$.

2. Since $A \subseteq A \cup A^{a^*}_m = aC^*_m(A)$, so $A \subseteq aC^*_m(A)$.

3. $aC^*_m(A) \cup aC^*_m(B) = (A \cup A^{a^*}_m) \cup (B \cup B^{a^*}_m) = (A \cup B) \cup (A^{a^*}_m \cup B^{a^*}_m) \subseteq (A \cup B) \cup (A \cup B)^{a*}_m = aC^*_m(A \cup B)$

4. $aC^*_m(A) = A \cup A^{a^*}_m$. By Theorem 3.2.5(6 and 7), we get that $A \cup A^{a^*}_m = (A \cup A^{a^*}_m) \cup ((A^{a^*}_m) \cup (A^{a^*}_m)^{a^*}_m) \subseteq (A \cup A^{a^*}_m) \cup (A \cup A^{a^*}_m)^{a^*}_m = aC^*_m(A \cup A^{a^*}_m)$. Therefore $aC^*_m(A) \subseteq aC^*_m(aC^*_m(A))$. □

Lemma 3.2.8. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A, B \subseteq X$. Then

1. If $A \subseteq B$, then $aC^*_m(A) \subseteq aC^*_m(B)$.
2. $aC^*_m(A \cap B) \subseteq aC^*_m(A) \cap aC^*_m(B)$.

Proof. 1. Assume that $A \subseteq B$. By Theorem 3.2.5(2), we get that $aC^*_m(A) = A \cup A^{a^*}_m \subseteq B \cup B^{a^*}_m = aC^*_m(B)$.

2. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by 1, we get that $aC^*_m(A \cap B) \subseteq aC^*_m(A) \cap aC^*_m(B)$. □

Definition 3.2.9. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A \subseteq X$. Then $\mathscr{M}^a$ is said to be *compatible with respect to* $\mathscr{I}$, denoted by $\mathscr{M}^a \sim \mathscr{I}$ if, for each $A \subseteq X$, $x \in A$ there exists $U \in \mathscr{M}^a(x)$ such that $U \cap A \in \mathscr{I}$, then $A \in \mathscr{I}$.

Theorem 3.2.10. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and A subset of X. Then the following are equivalent:

1. $\mathscr{M}^a \sim \mathscr{I}$.
2. If a subset A of X has a cover a-m-open sets of whose intersection with A is in $\mathscr{I}$, then A is in $\mathscr{I}$, in other words $A_m^{a^*} = \emptyset$, then $A \in \mathscr{I}$.
3. For every $A \subseteq X$, if $A \cap A_m^{a^*} = \emptyset$, $A \in \mathscr{I}$.
4. For every $A \subseteq X$, $A \setminus A_m^{a^*} \in \mathscr{I}$.
5. For every $A \subseteq X$, if A contains no nonempty subset B with $B \subseteq B_m^{a^*}$, then $A \in \mathscr{I}$.

Proof. (1) $\Longrightarrow$ (2) The proof is obvious.

(2) $\Longrightarrow$ (3) Let $A \subseteq X$ and $x \in A$. Then $x \notin A_m^{a^*}$ and there exists $U \in \mathscr{M}^a(x)$ such that $U \cap A \in \mathscr{I}$. Since A has a cover $A \subseteq \bigcup\{U \in \mathscr{M}^a(x) : x \in A\}$, by 2, $A \in \mathscr{I}$.

(3) $\Longrightarrow$ (4) Let $A \subseteq X$. We observe that $A \setminus A_m^{a^*} \subseteq A$ and by Theorem 3.2.5(2) $(A \setminus A_m^{a^*})_m^{a^*} \subseteq A_m^{a^*}$. So $(A \setminus A_m^{a^*})_m^{a^*} \cap (A \setminus A_m^{a^*}) \subseteq (A \setminus A_m^{a^*}) \cap A_m^{a^*} = \emptyset$. Then $(A \setminus A_m^{a^*})_m^{a^*} \cap (A \setminus A_m^{a^*}) = \emptyset$. By 3, then $A \setminus A_m^{a^*} \in \mathscr{I}$.

(4) $\Longrightarrow$ (5) Let $A \subseteq X$. By 4, then $A \setminus A_m^{a^*} \in \mathscr{I}$. We observe that $A = (A \setminus A_m^{a^*}) \cup (A \cap A_m^{a^*})$ and by Theorem 3.2.5(11), then $A_m^{a^*} = (A \cap A_m^{a^*})_m^{a^*}$. So, we get that $A_m^{a^*} \cap A = (A \cap A_m^{a^*})_m^{a^*} \cap A$, then $A_m^{a^*} \cap A \subseteq (A \cap A_m^{a^*})_m^{a^*}$. Since $(A \cap A_m^{a^*}) \subseteq A$, and by assumption $A \cap A_m^{a^*} = \emptyset$, then A contains no nonempty subset. Therefore $A \setminus A_m^{a^*} = A$, by 4, then $A \in \mathscr{I}$.

(5) $\Longrightarrow$ (1) Let $A \subseteq X$. Assume that for every $x \in A$, there exists $U \in \mathscr{M}^a(x)$ such that $U \cap A \in \mathscr{I}$. Then $A \cap A_m^{a^*} = \emptyset$. Since by Theorem 3.2.5(2), we get that $(A \setminus A_m^{a^*})_m^{a^*} \subseteq A_m^{a^*}$, then $(A \setminus A_m^{a^*})_m^{a^*} \cap (A \setminus A_m^{a^*}) \subseteq (A \setminus A_m^{a^*}) \cap A_m^{a^*} = \emptyset$. Next,

we will show that $A \setminus A_m^{a^*}$ contains no nonempty subset B with $B \subset B_m^{a^*}$. Suppose that $B \subseteq A \setminus A_m^{a^*}$ such that $B \subseteq B_m^{a^*}$ and $B \neq \emptyset$. Since $B \subseteq A \setminus A_m^{a^*}$, then $B_m^{a^*} \subseteq (A \setminus A_m^{a^*})_m^{a^*}$. Thus $B = B \cap B_m^{a^*} \subseteq (A \setminus A_m^{a^*}) \cap (A \cap A_m^{a^*})_m^{a^*} = \emptyset$. This is a contradiction. By 5, $A \setminus A_m^{a^*} \in \mathscr{I}$ and Hence $A = A \cap (X \setminus A_m^{a^*}) = A \setminus A_m^{a^*} \in \mathscr{I}$. □

Theorem 3.2.11. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and let $(A \cap A_m^{a^*})_m^{a^*} = A_m^{a^*}$ for all $A \subseteq X$. Then, for any $A \subseteq X$, $A \cap A_m^{a^*} = \emptyset$ implies that $A_m^{a^*} = \emptyset$.

Proof. Let $A \subseteq X$ be such that $A \cap A_m^{a^*} = \emptyset$. By using the assumption, we have $A_m^{a^*} = (A \cap A_m^{a^*})_m^{a^*} = (\emptyset)_m^{a^*} = \emptyset$. □

Theorem 3.2.12. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal, then the following properties are equivalent:

1. $\mathscr{M}^a \cap \mathscr{I} = \{\emptyset\}$.

2. If $J \in \mathscr{I}$, then $aI_m(J) = \emptyset$.

3. $X = X_m^{a^*}$.

Proof. (1) $\Longrightarrow$ (2) Let $\mathscr{M}^a \cap \mathscr{I} = \{\emptyset\}$ and $J \in \mathscr{I}$. Suppose that $x \in aI_m(J)$. Then there exists $U \in \mathscr{M}^a$ such that $x \in U \subseteq J$. Since $J \in \mathscr{I}$ and hence $\emptyset \neq \{x\} \subseteq U \in \mathscr{M}^a \cap \mathscr{I}$. This is a contradiction that $\mathscr{M}^a \cap \mathscr{I} = \{\emptyset\}$. Therefore $aI_m(J) = \emptyset$.

(2) $\Longrightarrow$ (3) Clearly $X_m^{a^*} \subseteq X$.

Suppose $x \in X$ and $x \notin X_m^{a^*}$. Then, there exists $U \in \mathscr{M}^a(x)$ such that $U \cap X \in \mathscr{I}$. Thus $U \in \mathscr{I}$. Since $U \in \mathscr{I}$ and $U \in \mathscr{M}^a(x)$, by 2, $U = aI_m(U) = \emptyset$. This is a contradiction, then $X \subseteq X_m^{a^*}$. Therefore $X = X_m^{a^*}$.

(3) $\Longrightarrow$ (1) Assume $\mathscr{M}^a \cap \mathscr{I} \neq \{\emptyset\}$. Then $U \in \mathscr{M}^a$ and $U \in \mathscr{I}$ such that $U \neq \emptyset$. Then there exists $x \in U \subseteq X$. Since $X = X_m^{a^*} = \{x \in X : V \cap X \notin \mathscr{I}$ for every $V \in \mathscr{M}^a(x)\}$. Then $U = U \cap X \notin \mathscr{I}$. Contradiction. Therefore $\mathscr{M}^a \cap \mathscr{I} = \{\emptyset\}$. □

3.3 On $\Re_m^a$-Operator In Minimal Structure Spaces with Ideals

In section, we shall introduce the operator $\Re_m^a$ in $(X, m, \mathscr{I})$. We shall discuss the behavior of this operator.

Definition 3.3.1. Let $(X, m, \mathscr{I})$ be a minimal structure with an ideal. An operator $\Re_m^a : P(X) \longrightarrow P(X)$ is defined as follows: for every $A \in P(X)$,

$$\Re_m^a(A) = \{x \in X : \text{there exists } U \in \mathscr{M}^a(x) \text{ such that } U \setminus A \in \mathscr{I}\}.$$

Theorem 3.3.2. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A \in P(X)$, then $\Re_m^a(A) = X \setminus (X \setminus A)_m^{a^*}$.

Proof. Let $x \in \Re_m^a(A)$, then there exists an a-m-open set U containing x such that $U \setminus A \in \mathscr{I}$. Thus $U \cap (X \setminus A) \in \mathscr{I}$. So $x \notin (X \setminus A)_m^{a^*}$ and hence $x \in X \setminus (X \setminus A)_m^{a^*}$. Therefore $\Re_m^a(A) \subseteq X \setminus (X \setminus A)_m^{a^*}$.

For the reverse inclusion, consider $x \in X \setminus (X \setminus A)_m^{a^*}$. Then $x \notin (X \setminus A)_m^{a^*}$. Thus there exists an a-m-open set U containing x such that $U \cap (X \setminus A) \in \mathscr{I}$. This implies that $U \setminus A \in \mathscr{I}$. Hence $x \in \Re_m^a(A)$. So $X \setminus (X \setminus A)_m^{a^*} \subseteq \Re_m^a(A)$.

Therefore $\Re_m^a(A) = X \setminus (X \setminus A)_m^{a^*}$. □

Example 3.3.3. Let $X = \{a, b, c, d\}$ with a minimal structure $m = \{\emptyset, \{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, X\}$ and $\mathscr{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, $A = \{a, b\}$. Then $\mathscr{M}^a = \{\emptyset, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$ and $\Re_m^a(A) = \{a, b\}$.

Theorem 3.3.4. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal. Then the following properties hold;

1. If $A \subseteq X$, then $\Re_m^a(A)$ is a-m-open.
2. If $A \subseteq B$, then $\Re_m^a(A) \subseteq \Re_m^a(B)$.
3. If $A, B \subseteq X$, then $\Re_m^a(A \cap B) \subseteq \Re_m^a(A) \cap \Re_m^a(B)$.
4. If $A, B \subseteq X$, then $\Re_m^a(A) \cup \Re_m^a(B) \subseteq \Re_m^a(A \cup B)$.
5. If $U \in \mathscr{M}^a$, then $U \subseteq \Re_m^a(U)$.
6. If $A \subseteq X$, then $\Re_m^a(A) \subseteq \Re_m^a(\Re_m^a(A))$.
7. If $A \subseteq X$, $U \in \mathscr{I}$, then $\Re_m^a(A \setminus U) = \Re_m^a(A)$.
8. If $A \subseteq X$, $U \in \mathscr{I}$, then $\Re_m^a(A \cup U) = \Re_m^a(A)$.

9. If $A \subseteq X$, then $\Re^a_m(A) = \Re^a_m(\Re^a_m(A))$ if and only if $(X \setminus A)^{a^*}_m = ((X \setminus A)^{a^*}_m)^{a^*}_m$.

10. If $(A \setminus B) \cup (B \setminus A) \in \mathscr{I}$, then $\Re^a_m(A) = \Re^a_m(B)$.

11. If $A \in \mathscr{I}$, then $\Re^a_m(A) = X \setminus X^{a^*}_m$.

Proof. 1. Let $A \subseteq X$, we know that $\Re^a_m(A) = X \setminus (X \setminus A)^{a^*}_m$ and $(X \setminus A)^{a^*}_m$ is a-m-closed. Therefore $\Re^a_m(A)$ is a-m-open.

2. Let $A \subseteq B$, then $X \setminus B \subseteq X \setminus A$. By Theorem 3.2.5(2), then $(X \setminus B)^{a^*}_m \subseteq (X \setminus A)^{a^*}_m$ and hence $X \setminus (X \setminus A)^{a^*}_m \subseteq X \setminus (X \setminus B)^{a^*}_m$. Therefore $\Re^a_m(A) \subseteq \Re^a_m(B)$.

3. Let $A, B \subseteq X$, then $A \cap B \subseteq A$ and $A \cap B \subseteq B$. So $\Re^a_m(A \cap B) \subseteq \Re^a_m(A)$ and $\Re^a_m(A \cap B) \subseteq \Re^a_m(B)$. Therefore $\Re^a_m(A \cap B) \subseteq \Re^a_m(A) \cap \Re^a_m(B)$.

4. Let $A, B \subseteq X$, hence $A \subseteq A \cup B$ and $B \subseteq A \cup B$. So $\Re^a_m(A) \subseteq \Re^a_m(A \cup B)$ and $\Re^a_m(B) \subseteq \Re^a_m(A \cup B)$. Therefore $\Re^a_m(A) \cup \Re^a_m(B) \subseteq \Re^a_m(A \cup B)$.

5. Assume that $U \in \mathscr{M}^a$. Then $X \setminus U$ is a-m-closed. By Theorem 3.2.5(4.), we get that $(X \setminus U)^{a^*}_m \subseteq aC_m(X \setminus U) = X \setminus U$. Therefore $U = X \setminus (X \setminus U) \subseteq X \setminus (X \setminus U)^{a^*}_m = \Re^a_m(U)$.

6. Let $A \subseteq X$. By 1, we get that $\Re^a_m(A)$ is a-m-open. By 5, we get that $\Re^a_m(A) \subseteq \Re^a_m(\Re^a_m(A))$.

7. Let $A \subseteq X$, $U \in \mathscr{I}$. By Theorem 3.2.5(11), we have $\Re^a_m(A \setminus U) = X \setminus (X \setminus (A \setminus U))^{a^*}_m = X \setminus ((X \setminus A) \cup U)^{a^*}_m = X \setminus (X \setminus A)^{a^*}_m$. Therefore $\Re^a_m(A \setminus U) = \Re^a_m(A)$.

8. Let $A \subseteq X$, $U \in \mathscr{I}$. By Theorem 3.2.5(12), we have $\Re^a_m(A \cup U) = X \setminus (X \setminus (A \cup U))^{a^*}_m = X \setminus ((X \setminus A) \setminus U)^{a^*}_m = X \setminus (X \setminus A)^{a^*}_m = \Re^a_m(A)$.

9. Let $A \subseteq X$, this follows from the facts;
i) $\Re^a_m(A) = X \setminus (X \setminus A)^{a^*}_m$ and
ii) $\Re^a_m(\Re^a_m(A)) = X \setminus [X \setminus (X \setminus (X \setminus A)^{a^*}_m)]^{a^*}_m = X \setminus ((X \setminus A)^{a^*}_m)^{a^*}_m$,
therefore $\Re^a_m(A) = \Re^a_m(\Re^a_m(A))$ if and only if $(X \setminus A)^{a^*}_m = ((X \setminus A)^{a^*}_m)^{a^*}_m$.

10. Let $(A \setminus B) \cup (B \setminus A) \in \mathscr{I}$ and $A \setminus B = J$, $B \setminus A = K$. So $J, K \in \mathscr{I}$ by heredity. We observe that $B = (A \setminus (A \setminus B)) \cup (B \setminus A) = (A \setminus J) \cup K$. Thus $\Re^a_m(A) = \Re^a_m(A \setminus J) = \Re^a_m((A \setminus J) \cup K) = \Re^a_m(B)$.

11. Let $A \in \mathscr{I}$. By Theorem 3.2.5(12), we get that $\Re^a_m(A) = X \setminus (X \setminus A)^{a^*}_m = X \setminus X^{a^*}_m$. □

Theorem 3.3.5. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A \subseteq X$. Then $\Re^a_m(A) = \bigcup\{U \in \mathscr{M}^a : U \setminus A \in \mathscr{I}\}$.

Proof. Let $H(A) = \bigcup\{U \in \mathscr{M}^a : U \setminus A \in \mathscr{I}\}$. We will to show that $H(A) = \Re^a_m(A)$. Assume that $x \in H(A)$. Then there exists $U \in \mathscr{M}^a(x)$ such that $U \setminus A \in \mathscr{I}$. By definition of $\Re^a_m$-operator, $x \in \Re^a_m(A)$. Let $x \in \Re^a_m(A)$, then there exists $U \in \mathscr{M}^a(x)$ such that $U \setminus A \in \mathscr{I}$. Since $U \in \mathscr{M}^a(x) \subseteq \mathscr{M}^a$, then $\Re^a_m(A) \subseteq H(A)$. □

Remark 3.3.6. Let $\mathscr{I} = \{\emptyset\}$. By Theorem 3.3.5, we get that $\Re^a_m(A) = \bigcup\{U \in \mathscr{M}^a : U \setminus A \in \{\emptyset\}\} = \bigcup\{U \in \mathscr{M}^a : U \subseteq A\} = aI_m(A)$, for any space (X, m).

Theorem 3.3.7. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A \subseteq X$. Then for any a-m-open set A of X, $\Re^a_m(A) = \cup\{U \in \mathscr{M} : (U \setminus A) \cup (A \setminus U) \in \mathscr{I}\}$.

Proof. $(\Longrightarrow)$ Let $H(A) = \bigcup\{U \in \mathscr{M}^a : (U \setminus A) \cup (A \setminus U) \in \mathscr{I}\}$. Since $\mathscr{I}$ is heredity, it is obvious that $H(A) = \bigcup\{U \in \mathscr{M}^a : (U \setminus A) \cup (A \setminus U) \in \mathscr{I}\} \subseteq \bigcup\{U \in \mathscr{M}^a : U \setminus A \in \mathscr{I}\} = \Re^a_m(A)$ for every $A \subseteq X$.

$(\Longleftarrow)$ Let $x \in \Re^a_m(A)$, then there exists $U \in \mathscr{M}^a(x)$ such that $U \setminus A \in \mathscr{I}$. Let $V = U \cup A \in \mathscr{M}^a$, then $(V \setminus A) \cup (A \setminus V) = U \setminus A \in \mathscr{I}$ and $x \in V \in \mathscr{M}^a$. Thus $x \in H(A)$. □

Theorem 3.3.8. Let $(X, m, \mathscr{I})$ be a minimal structure space an with ideal. If $\mathscr{M}^a \sim \mathscr{I}$, then $\Re^a_m(A) \setminus A \in \mathscr{I}$, for every $A \subseteq X$.

Proof. Assume that $\mathscr{M}^a \sim \mathscr{I}$ and $A \subseteq X$. If $\Re^a_m(A) \setminus A = \emptyset$, then $\Re^a_m \setminus A \in \mathscr{I}$. Assume that $\Re^a_m(A) \setminus A \neq \emptyset$, say $x \in \Re^a_m(A) \setminus A$, $x \in X \setminus (X \setminus A)^{a^*}_m$, then $x \notin (X \setminus A)^{a^*}_m$. Then there exists $U \in \mathscr{M}^a(x)$ such that $U \setminus A = U \cap (X \setminus A) \in \mathscr{I}$. We get that $x \in U \setminus A$. Then $(U \setminus A) \cap (\Re^a_m(A) \setminus A) \in \mathscr{I}$. So $U \cap (\Re^a_m(A) \setminus A) \in \mathscr{I}$. Therefore $\Re^a_m(A) \setminus A \in \mathscr{I}$. □

Proposition 3.3.9. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal, $\mathscr{M}^a \sim \mathscr{I}$ and $A \subseteq X$. If N is a nonempty a-m-open subset of $A^{a^*}_m \cap \Re^a_m(A)$, then $N \setminus A \in \mathscr{I}$ and $N \cap A \notin \mathscr{I}$.

Proof. Assume that $N \subseteq A_m^{a^*} \cap \Re_m^a(A)$ such that $N \in \mathscr{M}^a$ and $N \neq \emptyset$. Since $N \subseteq \Re_m^a(A)$, then $N \setminus A \subseteq \Re_m^a(A) \setminus A$. By Theorem 3.3.8 and heredity of $\mathscr{I}$, we get that $N \setminus A \in \mathscr{I}$. Since $N \in \mathscr{M}^a \setminus \{\emptyset\}$, $N \subseteq A_m^{a^*}$ and $A_m^{a^*} = \{x \in X : U \cap A \notin \mathscr{I}$ for every $U \in \mathscr{M}^a(x)\}$. Therefore $N \cap A \notin \mathscr{I}$. □

Corollary 3.3.10. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A \subseteq X$. If $\mathscr{M}^a \sim \mathscr{I}$, then $\Re_m^a(\Re_m^a(A)) = \Re_m^a(A)$.

Proof. Assume $\mathscr{M}^a \sim \mathscr{I}$ and $A \subseteq X$. We want to show that $\Re_m^a(\Re_m^a(A)) = \Re_m^a(A)$.

Clearly $\Re_m^a(A) \subseteq \Re_m^a(\Re_m^a(A))$.

Since

$$\begin{aligned}\Re_m^a(A) &\subseteq \Re_m^a(A) \cup A \\ &= (\Re_m^a(A) \cup A) \cap (A \cup (X \setminus A)) \\ &= A \cup (\Re_m^a(A) \cap (X \setminus A)) \\ &= A \cup (\Re_m^a(A) \setminus A),\end{aligned}$$

then $\Re_m^a(\Re_m^a(A)) \subseteq \Re_m^a(A \cup (\Re_m^a(A) \setminus A)) = \Re_m^a(A)$. (by Theorem 3.3.8 and 3.3.4(8)). Therefore $\Re_m^a(\Re_m^a(A)) = \Re_m^a(A)$. □

Theorem 3.3.11. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $\mathscr{M}^a \sim \mathscr{I}$. Then $\Re_m^a(A) = \bigcup\{\Re_m^a(U) : U \in \mathscr{M}^a, \Re_m^a(U) \setminus A \in \mathscr{I}\}$.

Proof. Let $H(A) = \bigcup\{\Re_m^a(U) : U \in \mathscr{M}^a, \Re_m^a(U) \setminus A \in \mathscr{I}\}$.

($\Longrightarrow$) Let $x \in H(A)$. Then there exists $x \in \Re_m^a(U)$ such that $U \in \mathscr{M}^a$ and $\Re_m^a(U) \setminus A \in \mathscr{I}$. By Theorem 3.3.4(1), we have $\Re_m^a(U) \in \mathscr{M}^a(x)$. So $x \in \Re_m^a(A)$. Therefore $H(A) \subseteq \Re_m^a(A)$.

($\Longleftarrow$) Let $x \in \Re_m^a(A)$. Then there exists $U \in \mathscr{M}^a(x)$ such that $U \setminus A \in \mathscr{I}$.

Since

$$\Re_m^a(U) \subseteq \Re_m^a(U) \cup (U \cap (X \setminus A))$$

and

$$X \setminus A \subseteq (X \setminus U) \cup (X \setminus A)$$

$$
\begin{aligned}
&= X \cap ((X \setminus U) \cup (X \setminus A)) \\
&= ((X \setminus U) \cup U) \cap ((X \setminus U) \cup (X \setminus A)) \\
&= (X \setminus U) \cup (U \cap (X \setminus A)),
\end{aligned}
$$

then

$$
\begin{aligned}
\Re_m^a(U) \setminus A &= \Re_m^a(U) \cap (X \setminus A) \\
&\subseteq \Big(\Re_m^a(U) \cup (U \cap (X \setminus A))\Big) \cap \Big((X \setminus U) \cup (U \cap (X \setminus A))\Big) \\
&= (\Re_m^a(U) \cap (X \setminus U)) \cup (U \cap (X \setminus A)) \\
&= (\Re_m^a(U) \setminus U) \cup (U \setminus A).
\end{aligned}
$$

By Theorem 3.3.8, $\Re_m^a(U) \setminus U \in \mathscr{I}$ and hence $U \setminus A \in \mathscr{I}$. Therefore $(\Re_m^a(U) \setminus U) \cup (U \setminus A) \in \mathscr{I}$ by property of $\mathscr{I}$. Then $\Re_m^a(U) \setminus A \in \mathscr{I}$. Since $x \in U$ and $U \subseteq \Re_m^a(U)$, then $x \in \Re_m^a(U)$. Then $x \in H(A)$. Therefore $\Re_m^a(A) \subseteq H(A)$. □

Definition 3.3.12. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A, B \subseteq X$. We define $A \bigodot B$ if $(A \setminus B) \cup (B \setminus A) \in \mathscr{I}$.

Theorem 3.3.13. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A, B \subseteq X$. Then $\bigodot$ is an equivalence relation.

Proof. I. Since $A \setminus A = \emptyset \in \mathscr{I}$, then $(A \setminus A) \cup (A \setminus A) \in \mathscr{I}$. Therefore $A \bigodot A$.

II. Assume $A \bigodot B$, then $(A \setminus B) \cup (B \setminus A) \in \mathscr{I}$. Since $(A \setminus B) \cup (B \setminus A) = (B \setminus A) \cup (A \setminus B) \in \mathscr{I}$, then $B \bigodot A$.

III. Assume $A \bigodot B$ and $B \bigodot C$, then $(A \setminus B) \cup (B \setminus A) \in \mathscr{I}$ and $(B \setminus C) \cup (C \setminus B) \in \mathscr{I}$. So $(A \setminus B) \cup (B \setminus C) \cup (B \setminus A) \cup (C \setminus B) \in \mathscr{I}$ by property of $\mathscr{I}$.

Since

$$
\begin{aligned}
A \setminus C &= A \cap (X \setminus C) \\
&\subseteq \Big(A \cup (B \cap (X \setminus C))\Big) \cap \Big((X \setminus B) \cup (B \cap (X \setminus C))\Big) \\
&= (A \cap (X \setminus B)) \cup (B \cap (X \setminus C)) \\
&= (A \setminus B) \cup (B \setminus C)
\end{aligned}
$$

and

$$\begin{aligned} C \setminus A &= C \cap (X \setminus A) \\ &\subseteq \Big(C \cup (B \cap (X \setminus A))\Big) \cap ((B \cap (X \setminus A)) \cup (X \setminus B)) \\ &= (B \cap (X \setminus A)) \cup (C \cap (X \setminus B)) \\ &= (B \setminus A) \cup (C \setminus B), \end{aligned}$$

then $(A \setminus C) \cup (C \setminus A) \subseteq ((A \setminus B) \cup (B \setminus C)) \cup ((B \setminus A) \cup (C \setminus B)) \in \mathscr{I}$. So $(A \setminus C) \cup (C \setminus A) \in \mathscr{I}$ by additive of $\mathscr{I}$. Therefore $A \bigodot C$.

From I, II and III, then $\bigodot$ is an equivalent relation. □

Theorem 3.3.14. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A, B \subseteq X$. If $A \bigodot B$ then $\Re^a_m(A) = \Re^a_m(B)$.

Proof. Assume $A \bigodot B$, then $(A \setminus B) \cup (B \setminus A) \in \mathscr{I}$. By Theorem 3.3.4(10), then $\Re^a_m(A) = \Re^a_m(B)$. □

Theorem 3.3.15. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal with $\mathscr{M}^a \sim \mathscr{I}$. If $U, V \in \mathscr{M}^a$ and $\Re^a_m(U) = \Re^a_m(V)$ then $U \bigodot V$.

Proof. Since $U \in \mathscr{M}^a$ and Theorem 3.3.4(5) $U \subseteq \Re^a_m(U)$ and hence $U \setminus V \subseteq \Re^a_m(U) \setminus V = \Re^a_m(V) \setminus V \in \mathscr{I}$ by Theorem 3.3.8 and 3.3.4(10). Then $U \setminus V \in \mathscr{I}$. Similarly $V \setminus U \subseteq \Re^a_m(V) \setminus U = \Re^a_m(U) \setminus U \in \mathscr{I}$. So $V \setminus U \in \mathscr{I}$. Then $(U \setminus V) \cup (V \setminus U) \in \mathscr{I}$ by additive of $\mathscr{I}$. Hence $U \bigodot V$. □

Theorem 3.3.16. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal with $\mathscr{M}^a \sim \mathscr{I}$. If $A, B \in \mathscr{B}$, and $\Re^a_m(A) = \Re^a_m(B)$ then $A \bigodot B$, where $\mathscr{B} = \{A \subseteq X :$ there exists $U \in \mathscr{M}^a$ such that $A \bigodot U\}$

Proof. Assume $A, B \in \mathscr{B}$, then there exists $U, V \in \mathscr{M}^a$ such that $A \bigodot U$ and $B \bigodot V$. By Theorem 3.3.14 $\Re^a_m(A) = \Re^a_m(U)$ and $\Re^a_m(B) = \Re^a_m(V)$. Since $\Re^a_m(A) = \Re^a_m(B)$, then $\Re^a_m(U) = \Re^a_m(V)$. By Theorem 3.3.16, we get that $U \bigodot V$. Therefore $A \bigodot B$ by transitivity of $\bigodot$. □

Proposition 3.3.17. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal. Then the following properties hold:

1. If $B \in \mathscr{B} \setminus \mathscr{I}$, then there exists $A \in \mathscr{M}^a \setminus \{\emptyset\}$ such that $B \bigodot A$.

2. If $\mathscr{M}^a \cap \mathscr{I} = \{\emptyset\}$, then $B \in \mathscr{B} \setminus \mathscr{I}$ if and only if there exists $A \in \mathscr{M}^a \setminus \{\emptyset\}$ such that $B \bigodot A$.

Proof. 1. Assume $B \in \mathscr{B} \setminus \mathscr{I}$, then $B \in \mathscr{B}$ and $B \notin \mathscr{I}$. So, there exists $A \in \mathscr{M}^a \setminus \{\emptyset\}$ such that $(B \setminus A) \cup (A \setminus B) \in \mathscr{I}$. Therefore $B \bigodot A$.

2. Let $\mathscr{M}^a \cap \mathscr{I} = \{\emptyset\}$. ($\Longrightarrow$) Assume $B \in \mathscr{B} \setminus \mathscr{I}$, by 1 then there exists $A \in \mathscr{M}^a$ such that $B \bigodot A$.

($\Longleftarrow$) Assume that there exists $A \in \mathscr{M}^a \setminus \{\emptyset\}$ such that $B \bigodot A$. Then $(B \setminus A) \cup (A \setminus B) \in \mathscr{I}$. So $B \in \mathscr{B}$. Next, we want to show that $B \notin \mathscr{I}$. Assume $B \in \mathscr{I}$. Let $J = B \setminus A$ and $K = A \setminus B$.

Since

$$\begin{aligned} B \setminus J &= (B \setminus (B \setminus A)) \\ &= \big(B \cap (X \setminus (B \cap (X \setminus A)))\big) \\ &= \big(B \cap ((X \setminus B) \cup A))\big) \\ &= (B \cap (X \setminus B)) \cup (B \cap A) \\ &= \emptyset \cup (B \cap A) \\ &= B \cap A, \end{aligned}$$

we get that

$$\begin{aligned} (B \setminus J) \cup K &= (B \cap A) \cup (A \setminus B) \\ &= (B \cap A) \cup (A \cap (X \setminus B)) \\ &= A \cup (B \cap (X \setminus B)) \\ &= A \cup \emptyset \\ &= A. \end{aligned}$$

Since $B \setminus J \subseteq B \in \mathscr{I}$ and $K = A \setminus B \in \mathscr{I}$, then $A = (B \setminus J) \cup K \in \mathscr{I}$. This is a contradiction that $\mathscr{M}^a \cap \mathscr{I} = \{\emptyset\}$. □

Theorem 3.3.18. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal. Then the following properties are equivalent:

1. $\mathscr{M}^a \cap \mathscr{I} = \{\emptyset\}$.

2. $\Re^a_m(\emptyset) = \emptyset$.

3. If $J \in \mathscr{I}$, then $\Re^a_m(J) = \emptyset$.

Proof. $(1) \Longrightarrow (2)$ Since $\mathscr{M}^a \cap \mathscr{I} = \{\emptyset\}$, then by Theorem 3.3.11 $\Re^a_m(\emptyset) = \bigcup\{U \in \mathscr{M}^a : U \in \mathscr{I}\} = \emptyset$.

$(2) \Longrightarrow (3)$ Assume that $J \in \mathscr{I}$. By Theorem 3.3.4(8) and 2, we get that $\Re^a_m(J) = \Re^a_m(\emptyset \cup J) = \Re^a_m(\emptyset) = \emptyset$. Therefore $\Re^a_m(J) = \emptyset$.

$(3) \Longrightarrow (1)$ Suppose $A \in \mathscr{M}^a \cap \mathscr{I}$, then $A \in \mathscr{I}$ and by 3, we get that $\Re^a_m(A) = \emptyset$. Since $A \in \mathscr{M}^a$, by Theorem 3.3.4(5), we get that $A \subseteq \Re^a_m(A) = \emptyset$. Hence $\mathscr{M}^a \cap \mathscr{I} = \{\emptyset\}$. □

Definition 3.3.19. A subset A in a minimal structure space with an ideal $(X, m, \mathscr{I})$ is said to be $\mathscr{I}^a_m$-*dense* if $A^{a^*}_m = X$.

Theorem 3.3.20. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal. Then for $x \in X$, $X \setminus \{x\}$ is $\mathscr{I}^a_m$-dense if and only if $\Re^a_m(\{x\}) = \emptyset$.

Proof. $(\Longrightarrow)$ Since $X = (X \setminus \{x\})^{a^*}_m$, then $\emptyset = X \setminus X = X \setminus (X \setminus \{x\})^{a^*}_m = \Re^a_m(\{x\})$.

$(\Longleftarrow)$ Since $X \setminus (X \setminus \{x\})^{a^*}_m = \Re^a_m(\{x\}) = \emptyset$, then $X = (X \setminus \{x\})^{a^*}_m$. □

3.4 a-$\mathscr{I}$-open set

Definition 3.4.1. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $G \subseteq X$. Then, G is called an a-$\mathscr{I}$-*open* set in X if $G = \emptyset$ or there exists a nonempty a-m-open K such that $K \setminus C^*_m(G) \in \mathscr{I}$ where $C^*_m(G) = G \cup G^*_m$ and $G^*_m = \{x \in X : G \cap U \notin \mathscr{I}$, for every $U \in m(x)\}$.

Example 3.4.2. Let $X = \{a, b, c, d\}$ with a minimal structure
$m = \{\emptyset, \{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, X\}$ and $\mathscr{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.
Then $\mathscr{M}^a = \{\emptyset, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.
Since $A^*_m = \{x \in X : A \cap U \notin \mathscr{I}$, for every $U \in m(x)\}$, $\emptyset^*_m = \emptyset$, $\{a\}^*_m = \emptyset$,

$\{b\}^*_m = \emptyset$, $\{c\}^*_m = \{c\}$, $\{d\}^*_m = \{d\}$, $\{a,b\}^*_m = \emptyset$, $\{a,c\}^*_m = \{c\}$, $\{a,d\}^*_m = \{d\}$, $\{b,c\}^*_m = \{c\}$, $\{b,d\}^*_m = \{d\}$, $\{c,d\}^*_m = \{c,d\}$, $\{a,b,c\}^*_m = \{c\}$, $\{a,b,d\}^*_m = \{d\}$, $\{a,c,d\}^*_m = \{c,d\}$, $\{b,c,d\}^*_m = \{c,d\}$, $X^*_m = X$.

Then, the class of all a-$\mathscr{I}$-open sets is $P(X)$.

Theorem 3.4.3. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $G \subseteq X$. For a-$\mathscr{I}$-open subset G_α in X for each $\alpha \in \Lambda$, then $\bigcup\{G_\alpha : \alpha \in \Lambda\}$ is a-$\mathscr{I}$-open subset of X.

Proof. Let G_α be an a-$\mathscr{I}$-open subset of X, for each $\alpha \in \Lambda$.

Case I, if $\bigcup\{G_\alpha; \alpha \in \Lambda\} = \emptyset$, then $\bigcup\{G_\alpha; \alpha \in \Lambda\}$ is a-$\mathscr{I}$-open subset of X.

Case II, if $\bigcup\{G_\alpha; \alpha \in \Lambda\} \neq \emptyset$, then there exists an $\alpha_i \in \Lambda$ such that $G_{\alpha_i} \neq \emptyset$. Since G_{α_i} is a-$\mathscr{I}$-open, then there exists a nonempty a-m-open subset K of X such that $K \setminus C^*_m(G_{\alpha_i}) \in \mathscr{I}$. Since $G_{\alpha_i} \subseteq \bigcup\{G_\alpha; \alpha \in \Lambda\}$, then $C^*_m(G_{\alpha_i}) \subseteq C^*_m(\bigcup\{G_\alpha; \alpha \in \Lambda\}$. So $K \setminus C^*_m(\bigcup\{G_\alpha; \alpha \in \Lambda\}) \subseteq K \setminus C^*_m(G_{\alpha_i})$. Since $K \setminus C^*_m(G_{\alpha_i}) \in \mathscr{I}$, then $K \setminus C^*_m(\bigcup\{G_\alpha; \alpha \in \Lambda\} \in \mathscr{I}$. Thus $\bigcup\{G_\alpha; \alpha \in \Lambda\}$ is a-$\mathscr{I}$-open of X. □

Lemma 3.4.4. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal. Then $\emptyset$, X are a-$\mathscr{I}$-open.

Proof. Clearly $\emptyset$ is a-$\mathscr{I}$-open set.

Since $C^*_m(X) = X \cup X^*_m = X$, X is a-m-open and $X \setminus C^*_m(X) = X \setminus X = \emptyset \in \mathscr{I}$, then X is a-$\mathscr{I}$-open. □

Lemma 3.4.5. Every a-m-open set is a-$\mathscr{I}$-open set.

Proof. Let G be a-m-open set.

Case I, $G = \emptyset$. So G is an a-$\mathscr{I}$-open set.

Case II, $G \neq \emptyset$. Then $C^*_m(G) = G \cup G^*_m$. Since $G \subseteq C^*_m(G)$ and G is a-m-open, then $G \setminus C^*_m(G) = \emptyset \in \mathscr{I}$. Therefore G is a-$\mathscr{I}$-open. □

Definition 3.4.6. The class of all a-$\mathscr{I}$-open sets in a minimal structure space with an ideal $(X, m, \mathscr{I})$ is denoted by $O\mathscr{I}_a(X)$. The *a-$\mathscr{I}$-interior* of A is denoted by $Int_{a\mathscr{I}}(A)$ is the union of all a-$\mathscr{I}$-open sets containing in A. A subset F of X is called *a-$\mathscr{I}$-closed* if its complement is a-$\mathscr{I}$-open in X. The class of all a-$\mathscr{I}$-closed sets in a minimal structure space with an ideal $(X, m, \mathscr{I})$ is denoted by $C\mathscr{I}_a(X)$. The *a-$\mathscr{I}$-closure* of a subset A of a minimal structure with an ideal $(X, m, \mathscr{I})$ is the intersection of all a-$\mathscr{I}$-closed sets containing A. The a-$\mathscr{I}$-closure of A is denoted by $Cl_{a\mathscr{I}}(A)$.

Remark 3.4.7. By Lemma 3.4.5, then every a-m-closed set is an a-$\mathscr{I}$-closed set.

Theorem 3.4.8. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A, B \subseteq X$. Then the following properties hold;

1. $A \subseteq Cl_{a\mathscr{I}}(A)$.
2. $Int_{a\mathscr{I}}(A) \subseteq A$.
3. If $A \subseteq B$, then $Cl_{a\mathscr{I}}(A) \subseteq Cl_{a\mathscr{I}}(B)$.
4. If $A \subseteq B$, then $Int_{a\mathscr{I}}(A) \subseteq Int_{a\mathscr{I}}(B)$
5. $Cl_{a\mathscr{I}}(A) \subseteq aC_m(A)$.
6. $aI_m(A) \subseteq Int_{a\mathscr{I}}(A)$.

Proof. 1. By definition $Cl_{a\mathscr{I}}(A)$, then $A \subseteq Cl_{a\mathscr{I}}(A)$.

2. By definition $Int_{a\mathscr{I}}(A)$, then $Int_{a\mathscr{I}}(A) \subseteq A$.
3. Suppose that $A \subseteq B$ and $x \notin Cl_{a\mathscr{I}}(B)$. Then, there exists an a-$\mathscr{I}$-closed set F such that $B \subseteq F$ and $x \notin F$. It follows from $A \subseteq B$ that $x \notin Cl_{a\mathscr{I}}(A)$.
4. Let $A \subseteq B$, and suppose that $x \in Int_{a\mathscr{I}}(A)$. Then , there exists an a-$\mathscr{I}$-open set U such that $U \subseteq A$ and $x \in U$. Since $A \subseteq B$, we get that $x \in Int_{a\mathscr{I}}(B)$.
5. Assume that $x \notin aC_m(A)$. Then, there exists a-m-closed F in X such that $A \subseteq F$ and $x \notin F$. Since every a-m-closed is a-$\mathscr{I}$-closed, we get that F is a-$\mathscr{I}$-closed. Therefore $x \notin Cl_{aI}(A)$.
6. Assume that $x \in aI_m(A)$. Then, there exists U such that $U \subseteq A$ and $x \in U$. Since every a-m-open is a-$\mathscr{I}$-open, then a-m-open U is a-$\mathscr{I}$-open. Therefore $x \in Int_{a\mathscr{I}}(A)$. □

Theorem 3.4.9. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A \subseteq X$. Then $x \in Cl_{a\mathscr{I}}(A)$ if and only if $V \cap A \neq \emptyset$ for all $V \in O\mathscr{I}_a(X)$ containing x.

Proof. $(\Longrightarrow)$ Assume $V \cap A = \emptyset$ for some $V \in O\mathscr{I}_a(X)$ containing x. Then $A \subseteq X \setminus V$. Since V is a-$\mathscr{I}$-open, then $X \setminus V$ is a-$\mathscr{I}$-closed, and not containing x. We get that $x \notin Cl_{a\mathscr{I}}(A)$. Therefore, if $x \in Cl_{a\mathscr{I}}(A)$ then $V \cap A \neq \emptyset$ for all $V \in O\mathscr{I}_a(X)$ containing x.

$(\Longleftarrow)$ Assume $x \notin Cl_{a\mathscr{I}}(A)$. Then, there exists a-$\mathscr{I}$-closed F such that $A \subseteq F$ and $x \notin F$. Then $X \setminus F$ is a-$\mathscr{I}$-open and contain x. We get that $(X \setminus F) \cap A = \emptyset$. Therefore, if $V \cap A \neq \emptyset$ for all $V \in O\mathscr{I}_a(X)$ containing x, then $x \in Cl_{a\mathscr{I}}(A)$. □

Theorem 3.4.10. Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A \subseteq X$. Then the following properties hold;

1. $Cl_{a\mathscr{I}}(A) = X \setminus Int_{a\mathscr{I}}(X \setminus A)$.
2. $Int_{a\mathscr{I}}(A) = X \setminus Cl_{a\mathscr{I}}(X \setminus A)$.

Proof. 1. Assume that $x \notin X \setminus Int_{a\mathscr{I}}(X \setminus A)$. Then $x \in Int_{a\mathscr{I}}(X \setminus A)$. So, there exists $V \in O\mathscr{I}_a(X)$ such that $x \in V \subseteq X \setminus A$. Then $V \cap A = \emptyset$. By Theorem 3.4.9, we get that $x \notin Cl_{a\mathscr{I}}(A)$. Therefore $Cl_{a\mathscr{I}}(A) \subseteq X \setminus Int_{a\mathscr{I}}(X \setminus A)$. Next, we want to show that $X \setminus Int_{a\mathscr{I}}(X \setminus A) \subseteq Cl_{a\mathscr{I}}(A)$. Assume that $x \notin Cl_{a\mathscr{I}}(A)$. Then, there exists $V \in O\mathscr{I}_a(X)$ containing x such that $V \cap A = \emptyset$. So $V \subseteq X \setminus A$. Then $x \in Int_{a\mathscr{I}}(X \setminus A)$. We get that $x \notin X \setminus Int_{a\mathscr{I}}(X \setminus A)$. Then $X \setminus Int_{a\mathscr{I}}(X \setminus A) \subseteq Cl_{a\mathscr{I}}(A)$. Therefore $Cl_{a\mathscr{I}}(A) = X \setminus Int_{a\mathscr{I}}(X \setminus A)$.

2. Since $X \setminus A \subseteq X$, by 1, we get that $Cl_{a\mathscr{I}}(X \setminus A) = X \setminus Int_{a\mathscr{I}}(X \setminus (X \setminus A))$. Then $Cl_{a\mathscr{I}}(X \setminus A) = X \setminus Int_{a\mathscr{I}}(A)$. This implies that $Int_{a\mathscr{I}}(A) = X \setminus Cl_{a\mathscr{I}}(X \setminus A)$. □

Definition 3.4.11. A mapping $f : (X, m_X, \mathscr{I}) \longrightarrow (Y, m_Y)$ is said to be a-$\mathscr{I}$-*continuous* if for each $x \in X$ and each m_Y-open K of Y containing $f(x)$, there exists $G \in O\mathscr{I}_a(X)$ containing x such that $f(G) \subseteq K$.

Theorem 3.4.12. For a mapping $f : (X, m_X, \mathscr{I}) \longrightarrow (Y, m_Y)$, the following properties are equivalent:

1. f is a-$\mathscr{I}$-continuous.
2. $f^{-1}(K)$ is a-$\mathscr{I}$-open in X for each m_Y-open set K of Y.
3. $f^{-1}(F)$ is a-$\mathscr{I}$-closed in X for each m_Y-closed set F of Y.
4. $f(Cl_{a\mathscr{I}}(A)) \subseteq Cl_{m_Y}(f(A))$ for each subset A of X.
5. $Cl_{a\mathscr{I}}(f^{-1}(B)) \subseteq f^{-1}(Cl_{m_Y}(B))$ for each subset B of Y.
6. $f^{-1}(Int_{m_Y}(B)) \subseteq Int_{a\mathscr{I}}(f^{-1}(B))$ for each subset B of Y.

Proof. (1) $\Longrightarrow$ (2) Let K be any m_Y-open subset of Y and $x \in f^{-1}(K)$. Then, there exists $G_x \in O\mathscr{I}_a(X)$ containing x such that $f(G_x) \subseteq K$. We get that $x \in G \subseteq f^{-1}(f(G_x)) \subseteq f^{-1}(K)$. Thus $f^{-1}(K) = \bigcup_{x\in f^{-1}(K)} G_x$. By Theorem 3.4.3, we get that $f^{-1}(K)$ is a-$\mathscr{I}$-open in X.

(2) $\Longrightarrow$ (3) Let F be m_Y-closed in Y, we get that $Y \setminus F$ is m_Y-open in Y. By 2, we have $f^{-1}(Y \setminus F)$ is a-$\mathscr{I}$-open in X. But $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$. Therefore $f^{-1}(F)$ is a-$\mathscr{I}$-closed in X.

(3) $\Longrightarrow$ (1) Let $x \in X$ and K be m_Y-open in Y such that $f(x) \in K$. Then $Y\setminus K$ is a-$\mathscr{I}$-closed. By 3, we get that $f^{-1}(Y\setminus K)$ is a-$\mathscr{I}$-closed. So $f^{-1}(K) = X\setminus f^{-1}(Y\setminus K)$ is a-$\mathscr{I}$-open, setting $G := f^{-1}(K)$. Now, $f(G) = f(f^{-1}(K)) \subseteq K$ and $x \in G$. Therefore f is a-$\mathscr{I}$-continuous.

(1) $\Longrightarrow$ (6) Let $B \subseteq Y$ and $x \in f^{-1}(Int_{m_Y}(B))$. Then, there exists $y \in Int_{m_Y}(B)$ such that $f(x) = y$. Since $y \in Int_{m_Y}(B)$, there exists m_Y-open G such that $y \in G \subseteq B$. Then $f(x) = y \in G$. By 1, there exists a-$\mathscr{I}$-open U of X such that $x \in U$ and $f(U) \subseteq G$. So $U \subseteq f^{-1}(f(U)) \subseteq f^{-1}(G) \subseteq f^{-1}(B)$. This implies that $x \in Int_{a\mathscr{I}}(f^{-1}(B))$. Therefore $f^{-1}(Int_{m_Y}(B)) \subseteq Int_{a\mathscr{I}}(f^{-1}(B))$.

(6) $\Longrightarrow$ (1) Let $x \in X$, and K be m_Y-open in Y such that $f(x) \in K$. By 6, we get that $f^{-1}(K) = f^{-1}(Int_{m_Y}(K)) \subseteq Int_{a\mathscr{I}}(f^{-1}(K)) \subseteq f^{-1}(K)$. Then $f^{-1}(K) = Int_{a\mathscr{I}}(f^{-1}(K))$. Now, $G := f^{-1}(K)$ is a-$\mathscr{I}$-open and $x \in f^{-1}(K)$. We get that $f(G) = f(f^{-1}(K)) \subseteq K$. Therefore f is a-$\mathscr{I}$-continuous.

(6) $\Longrightarrow$ (4) Let $A \subseteq X$. By 6, we get that $f^{-1}(Int_{m_Y}(Y\setminus f(A))) \subseteq Int_{a\mathscr{I}}(f^{-1}(Y\setminus f(A))$.

Then

$$
\begin{aligned}
X \setminus f^{-1}(Cl_{m_Y}(f(A))) &= f^{-1}(Y \setminus Cl_{m_Y}(f(A))) \\
&= f^{-1}(Int_{m_Y}(Y \setminus f(A))) \\
&\subseteq Int_{a\mathscr{I}}(f^{-1}(Y \setminus f(A)) \\
&= Int_{a\mathscr{I}}(X \setminus f^{-1}(f(A))) \\
&= X \setminus Cl_{a\mathscr{I}}(f^{-1}(f(A))).
\end{aligned}
$$

So, we have $Cl_{a\mathscr{I}}(A) \subseteq Cl_{a\mathscr{I}}(f^{-1}(f(A))) \subseteq f^{-1}(Cl_{m_Y}(f(A))$. Therefore $f(Cl_{a\mathscr{I}}(A)) \subseteq f(f^{-1}(Cl_{m_Y}(f(A))) \subseteq Cl_{m_Y}(f(A))$.

(4) $\Longrightarrow$ (5) Let $B \subseteq Y$. By 4, we get that $f(Cl_{a\mathscr{I}}(f^{-1}(B)) \subseteq Cl_{m_Y}(f(f^{-1}(B)) \subseteq Cl_{m_Y}(B)$. Therefore $Cl_{a\mathscr{I}}(f^{-1}(B)) \subseteq f^{-1}(f(Cl_{a\mathscr{I}}(f^{-1}(B))) \subseteq f^{-1}(Cl_{m_Y}(B))$.

(5) $\Longrightarrow$ (6) Let $B \subseteq Y$. By 5, we get that $Cl_{a\mathscr{I}}(f^{-1}(Y \setminus B)) \subseteq f^{-1}(Cl_{m_Y}(Y \setminus B))$. Then

$$
\begin{aligned}
X \setminus Int_{a\mathscr{I}}(f^{-1}(B)) &= Cl_{a\mathscr{I}}(X \setminus f^{-1}(B) \\
&= Cl_{a\mathscr{I}}(f^{-1}(Y \setminus B)) \\
&\subseteq f^{-1}(Cl_{m_Y}(Y \setminus B)) \\
&= f^{-1}(Y \setminus Int_{m_Y}(B)) \\
&= X \setminus f^{-1}(Int_{m_Y}(B))
\end{aligned}
$$

Therefore $f^{-1}(Int_{m_Y}(B)) \subseteq Int_{a\mathscr{I}}(f^{-1}(B))$. □

CHAPTER 4

Conclusions

The aim of this thesis is to introduce the results of properties of some sets in a minimal structure space with an ideal. And we study some properties of δ-m-open sets, a-m-open sets in a minimal structure space with an ideal are introduced. Moreover we have defined a-m-local function and $\Re_m^a$-operator in a minimal structure space with an ideal. Some properties of their are obtained. Moreover, we introduce the notions a-$\mathscr{I}$-open sets and a-$\mathscr{I}$-continuous. The results are as follows:

1) Let (X, m) be a minimal structure space and $A, B \subseteq X$. The following property hold;

    1.1) If $A \subseteq B$, then $aC_m(A) \subseteq aC_m(B)$.

    1.2) If $A \subseteq B$, then $aI_m(A) \subseteq aI_m(B)$.

2) Let (X, m) be a minimal structure space and $A \subseteq X$. The following property hold;

    2.1) $aC_m(aC_m(A)) = aC_m(A)$.

    2.2) $aI_m(aI_m(A)) = aI_m(A)$.

3) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A, B \subseteq X$. The following properties hold;

    3.1) $(\emptyset)_m^{a^*} = \emptyset$.

    3.2) If $A \subseteq B$, then $A_m^{a^*} \subseteq B_m^{a^*}$.

    3.3) For another ideal $\mathscr{J}$ on X such that $\mathscr{I} \subseteq \mathscr{J}$ then $A_m^{a^*}(\mathscr{J}, m) \subseteq A_m^{a^*}(\mathscr{I}, m)$.

    3.4) $A_m^{a^*} \subseteq aC_m(A)$.

    3.5) $A_m^{a^*} = aC_m(A_m^{a^*})$, (i.e., $A_m^{a^*}$ is an a-m-closed subset).

    3.6) $(A_m^{a^*})_m^{a^*} \subseteq A_m^{a^*}$.

3.7) $A_m^{a^*} \cup B_m^{a^*} \subseteq (A \cup B)_m^{a^*}$.

3.8) $(A \cap B)_m^{a^*} \subseteq A_m^{a^*} \cap B_m^{a^*}$.

3.9) $(A \setminus B)_m^{a^*} \setminus B_m^{a^*} \subseteq A_m^{a^*} \setminus B_m^{a^*}$.

3.10) If $A \in \mathscr{I}$, then $A_m^{a^*} = \emptyset$.

3.11) If $U \in \mathscr{I}$, then $A_m^{a^*} = (A \cup U)_m^{a^*}$.

3.12) If $U \in \mathscr{I}$, then $A_m^{a^*} = (A \setminus U)_m^{a^*}$.

4) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal where $aC_m^*(A) = A \cup A_m^{a^*}$ and $A, B \subseteq X$. Then

4.1) $aC_m^*(\emptyset) = \emptyset$.

4.2) $A \subseteq aC_m^*(A)$.

4.3) $aC_m^*(A) \cup aC_m^*(B) \subseteq aC_m^*(A \cup B)$.

4.4) $aC_m^*(A) \subseteq aC_m^*(aC_m^*(A))$.

5) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A, B \subseteq X$. Then

5.1) If $A \subseteq B$, then $aC_m^*(A) \subseteq aC_m^*(B)$.

5.2) $aC_m^*(A \cap B) \subseteq aC_m^*(A) \cap aC_m^*(B)$.

6) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and A subset of X. Then the following are equivalent:

6.1) $\mathscr{M}^a \sim \mathscr{I}$.

6.2) If a subset A of X has a cover a-m-open sets of whose intersection with A is in $\mathscr{I}$, then A is in $\mathscr{I}$, in other words $A_m^{a^*} = \emptyset$, then $A \in \mathscr{I}$.

6.3) For every $A \subseteq X$, if $A \cap A_m^{a^*} = \emptyset$, $A \in \mathscr{I}$.

6.4) For every $A \subseteq X$, $A \setminus A_m^{a^*} \in \mathscr{I}$.

6.5) For every $A \subseteq X$, if A contains no nonempty subset B with $B \subseteq B_m^{a^*}$, then $A \in \mathscr{I}$.

7) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal, then the following properties are equivalent:

7.1) $\mathscr{M}^a \cap \mathscr{I} = \{\emptyset\}$.

7.2) If $J \in \mathscr{I}$, then $aI_m(J) = \emptyset$.

7.3) $X = X_m^{a^*}$.

8) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A \in P(X)$, then $\Re_m^a(A) = X \setminus (X \setminus A)_m^{a^*}$.

9) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal. Then the following properties hold;

9.1) If $A \subseteq X$, then $\Re_m^a(A)$ is a-m-open.

9.2) If $A \subseteq B$, then $\Re_m^a(A) \subseteq \Re_m^a(B)$.

9.3) If $A, B \subseteq X$, then $\Re_m^a(A \cap B) \subseteq \Re_m^a(A) \cap \Re_m^a(B)$.

9.4) If $A, B \subseteq X$, then $\Re_m^a(A) \cup \Re_m^a(B) \subseteq \Re_m^a(A \cup B)$.

9.5) If $U \in \mathscr{M}^a$, then $U \subseteq \Re_m^a(U)$.

9.6) If $A \subseteq X$, then $\Re_m^a(A) \subseteq \Re_m^a(\Re_m^a(A))$.

9.7) If $A \subseteq X$, $U \in \mathscr{I}$, then $\Re_m^a(A \setminus U) = \Re_m^a(A)$.

9.8) If $A \subseteq X$, $U \in \mathscr{I}$, then $\Re_m^a(A \cup U) = \Re_m^a(A)$.

9.9) If $A \subseteq X$, then $\Re_m^a(A) = \Re_m^a(\Re_m^a(A))$ if and only if $(X \setminus A)_m^{a^*} = ((X \setminus A)_m^{a^*})_m^{a^*}$.

9.10) If $(A \setminus B) \cup (B \setminus A) \in \mathscr{I}$, then $\Re_m^a(A) = \Re_m^a(B)$.

9.11) If $A \in \mathscr{I}$, then $\Re_m^a(A) = X \setminus X_m^{a^*}$.

10) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A \subseteq X$. Then $\Re_m^a(A) = \bigcup\{U \in \mathscr{M}^a : U \setminus A \in \mathscr{I}\}$.

11) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A \subseteq X$. Then for any a-m-open set A of X, $\Re_m^a(A) = \cup\{U \in \mathscr{M} : (U \setminus A) \cup (A \setminus U) \in \mathscr{I}\}$.

12) Let $(X, m, \mathscr{I})$ be a minimal structure space an with ideal. If $\mathscr{M}^a \sim \mathscr{I}$, then $\Re_m^a(A) \setminus A \in \mathscr{I}$, for every $A \subseteq X$.

13) Let $(X, m, \mathscr{I})$ be a minimal structure space an with ideal, $\mathscr{M}^a \sim \mathscr{I}$ and $A \subseteq X$. If N is a nonempty a-m-open subset of $A_m^{a^*} \cap \Re_m^a(A)$, then $N \setminus A \in \mathscr{I}$ and $N \cap A \notin \mathscr{I}$.

14) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A \subseteq X$. If $\mathscr{M}^a \sim \mathscr{I}$ then, $\Re_m^a(\Re_m^a(A)) = \Re_m^a(A)$.

15) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $\mathscr{M}^a \sim \mathscr{I}$. Then $\Re_m^a(A) = \bigcup\{\Re_m^a(U) : U \in \mathscr{M}^a, \Re_m^a(U) \setminus A \in \mathscr{I}\}$.

16) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal. Then the following properties are equivalent:

16.1) $\mathscr{M}^a \cap \mathscr{I} = \{\emptyset\}$.

16.2) $\Re_m^a(\emptyset) = \emptyset$.

16.3) If $J \in \mathscr{I}$, then $\Re_m^a(J) = \emptyset$.

17) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A, B \subseteq X$. Then the following properties hold;

17.1) $A \subseteq Cl_{a\mathscr{I}}(A)$.

17.2) $Int_{a\mathscr{I}}(A) \subseteq A$.

17.3) If $A \subseteq B$, then $Cl_{a\mathscr{I}}(A) \subseteq Cl_{a\mathscr{I}}(B)$.

17.4) If $A \subseteq B$, then $Int_{a\mathscr{I}}(A) \subseteq Int_{a\mathscr{I}}(B)$.

17.5) $Cl_{a\mathscr{I}}(A) \subseteq aC_m(A)$.

17.6) $aI_m(A) \subseteq Int_{a\mathscr{I}}(A)$.

18) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A \subseteq X$. Then $x \in Cl_{a\mathscr{I}}(A)$ if and only if $V \cap A \neq \emptyset$ for all $V \in O\mathscr{I}_a(X)$ containing x.

19) Let $(X, m, \mathscr{I})$ be a minimal structure space with an ideal and $A \subseteq X$. Then the following properties hold;

19.1) $Cl_{a\mathscr{I}}(A) = X \setminus Int_{a\mathscr{I}}(X \setminus A)$.

19.2) $Int_{a\mathscr{I}}(A) = X \setminus Cl_{a\mathscr{I}}(X \setminus A)$.

20) For a mapping $f : (X, m_X, \mathscr{I}) \longrightarrow (Y, m_Y)$, the following properties are equivalent:

20.1) f is a-$\mathscr{I}$-continuous.

20.2) $f^{-1}(K)$ is a-$\mathscr{I}$-open in X for each m_Y-open set K of Y.

20.3) $f^{-1}(F)$ is a-$\mathscr{I}$-closed in X for each m_Y-closed set F of Y.

20.4) $f(Cl_{a\mathscr{I}}(A)) \subseteq Cl_{m_Y}(f(A))$ for each subset A of X.

20.5) $Cl_{a\mathscr{I}}(f^{-1}(B)) \subseteq f^{-1}(Cl_{m_Y}(B))$ for each subset B of Y.

20.6) $f^{-1}(Int_{m_Y}(B)) \subseteq Int_{a\mathscr{I}}(f^{-1}(B))$ for each subset B of Y.

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BIOGRAPHY

BIOGRAPHY

Name	Mr. Yutthapong Manuttiparom
Date of birth	July 28, 1978
Place of birth	Nongkhai Province, Thailand
Institution attended	
1996	High School in Pathumthepwitthayakarn School, Nongkhai, Thailand
2008	Bachelor of Business Administrator Program in Finance and Banking, Ramkhamhaeng University, Thailand
2010	Master of Industrial Education and Technology, King Mongkut's University of Technology Thonburi, Thailand
2015	Bachelor of Science in Mathematics, Buriram Rajabhat University, Thailand
2020	Master of Science in Mathematics, Mahasarakham University, Thailand
Contact address	
	719, Meechai Road, Mueang Nongkhai district, Nongkhai provice 43000, Thailand
	59010283013@msu.ac.th